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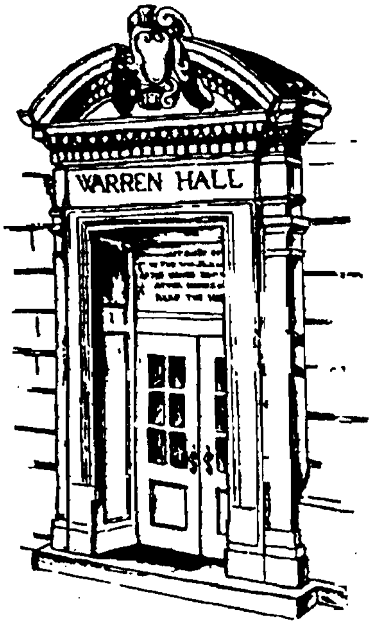

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A NEW ALGORITHM FOR COMPUTING COMPENSATED INCOME FROM ORDINARY DEMAND FUNCTIONS

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Abstract

This paper proposes a **RE**versible Second-**OR**der Taylor (RESORT) expansion of the expenditure function to compute compensated income from ordinary demand functions as an alternative to the algorithm proposed by Vartia. These algorithms provide measures of Hicksian welfare changes and Konüs-type cost of living indices. RESORT also validates the results by checking the matrix of compensated price effects, obtained through the Slutsky equation, for symmetry and negative semi-definiteness as required by expenditure minimization. In contrast, Vartia's algorithm provides no validation procedure. RESORT is similar to Vartia's algorithm in using price steps. It computes compensated income at each step "forward" from the initial to the terminal prices, and insures that the compensated income computed "backward" is equal to its value computed in the "forward" procedure. Thus, RESORT is "reversible" and guarantees unique values of compensated income for each set of prices and, as a result, also unique measures of welfare changes and cost of living indices. These unique results are not, however, guaranteed by the usual Taylor series expansion for computing compensated income.

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1. Introduction

The existence in theory of the income compensation function was established by Hurwicz and Uzawa (1971). Chipman and Moore (1980) noted that what was then lacking was a "practical algorithm" for computing compensated income from ordinary demand functions. In theory, the significance of this algorithm lies in permitting compensated income, which is unobservable, to be obtained from ordinary demand functions estimated from price and income data. In practice, the value of this algorithm lies in providing at the same time measures of Hicksian welfare changes and Konüs-type true indices of the cost of living when prices change. Moreover, this algorithm widens the scope of empirical demand analysis to demand systems with unknown underlying utility functions that in principle are "integrable."[1] Indeed, it is precisely for these demand systems that a compensated income "approximation" algorithm is necessary. Approximations are unnecessary if the utility function is known because the "true" compensated income is obtainable exactly from the dual expenditure function. Thus, demand systems need not be limited to the standard models derived from an indirect utility function or expenditure function of an explicit parametric form such as the translog (Christensen, Jorgenson, and Lau, 1975; Christensen and Caves, 1980) and the "almost ideal demand system" or AIDS (Deaton and Muellbauer, 1980a and 1980b).

[1]Integrability implies a well-behaved demand system for which an underlying utility function exists. The utility function may not, however, be recoverable in closed form and, thus, may remain unknown. An "integrable" demand system has a symmetric and negative semi-definite matrix of compensated price effects. This is equivalent to linear homogeneity and concavity in prices of the underlying expenditure function that may also be unknown. An example is the generalized logit demand model discussed in section 5.

To fill the need noted earlier by Chipman and Moore, Vartia (1983) proposed numerical integration of ordinary demand functions to compute compensated income. The procedure is implemented stepwise, over price increments connecting the initial to the terminal price vector. This paper, however, proposes a **RE**versible Second-**OR**der **T**aylor (RESORT) expansion of the expenditure function to compute compensated income from ordinary demand functions. It follows Vartia's stepwise procedure and computes compensated income at each price step "forward," starting with the initial income and prices. It is "reversible" in that the solution of compensated income at any intermediate price step will be the same when the procedure is reversed "backward," starting with the terminal compensated income and prices. That is, RESORT yields a unique compensated income at each price step. This is a desirable property because a unique solution is not assured by a standard Taylor series approximation. Also, the stepwise procedure reduces the approximation error of a one-step second-order Taylor expansion.

The second-order terms of RESORT yield the matrix of compensated price effects. All the elements of this matrix are computed from ordinary demand functions by means of the Slutsky equation. Thus, RESORT provides a check for the validity of the compensated income computed at each price step by evaluating the above matrix for symmetry and negative semi-definiteness as required by expenditure minimization. In contrast, Vartia's algorithm provides no check for the validity of the computed compensated income because his procedure does not compute the matrix of compensated price effects. It is important, however, to provide validity checks in empirical applications because there is no assurance that the demand system utilized by the algorithm is well-behaved within the price range under examination.

Section 2 of this paper shows that measuring compensated incomes when prices change yields measures of welfare changes and cost of living indices. Section 3 presents the RESORT algorithm while section 4 presents Vartia's algorithm. Section 5 presents numerical results from the computations of compensated income in four separate illustrations. While these algorithms are applicable to any arbitrary finite number of goods, the illustrations utilize only two goods for simplicity. The first three illustrations use ordinary demand functions derived from an explicit utility function. In these illustrations, the "true" compensated incomes are known. Therefore, it is possible to compare the closeness to the true compensated income of the approximations from RESORT and Vartia's algorithm. Vartia's and the RESORT approximations are both very close to the true values but those from RESORT are closer in two of the three cases. The fourth illustration utilizes ordinary demand functions from a model with an unknown utility function. This is precisely the situation for which Vartia's method and RESORT are intended. In this case, Vartia's and the RESORT approximations are very close to each other but their relative precision cannot be assessed because the true compensated income is unknown.

In one of the four illustrations, negative semi-definiteness of the matrix of compensated price effects was violated to demonstrate the ability of RESORT to detect violations of theoretical restrictions. It is shown that RESORT can determine when demand systems violate the requirements of expenditure minimization, at the same instance when the violations would be undetected by Vartia's algorithm.

Section 6 concludes this paper.

2. Compensated Income, Welfare Change, and the Cost of Living Index

Consider a situation where prices change from p^0 to p^T while utility correspondingly changes from U^0 to U^T. Assume that there are n goods so that $p^0 = \{p_i^0\}$ and $p^T = \{p_i^T\}$, $i = 1, 2, \ldots, n$. In general, let $C(p, U)$ be an expenditure function that gives the minimum cost at prices p to attain a given utility level U where p is either p^0 or p^T and U is either U^0 or U^T. The superscript "0" denotes the original (old) and "T" denotes the terminal (new) situations.

By duality between expenditure minimization and utility maximization, $C(p^0, U^0)$ and $C(p^T, U^T)$ are equal to the actual total expenditures on goods bought at the prices given by the vectors p^0 and p^T. These expenditures are known, given by the ordinary demand functions estimated from observable price, income and quantity data. However, $C(p^T, U^0)$ and $C(p^0, U^T)$ are not observable because these are defined by compensated demand functions that cannot be estimated directly since utility is not observable. The problem is to approximate $C(p^T, U^0)$ given $C(p^0, U^0)$ or to approximate $C(p^0, U^T)$ given $C(p^T, U^T)$ based only on the ordinary demand functions that give these starting values. The underlying utility function is presumed unknowable.

By definition, $C(p^T, U^0)$ is the "compensated income" that allows the consumer to maintain the utility level U^0 as prices change from p^0 to p^T. Similarly, $C(p^0, U^T)$ is the "compensated income" that allows the consumer to maintain the utility level U^T as prices change back from p^T to p^0. The *change* in compensated income between these price situations determines Hicksian measures of welfare change while the *ratio* of compensated incomes gives Konüs-type true indices of the cost of living. The Hicksian compensating variation (CV) and equivalent variation (EV) are defined as,

(1) $$CV = C(p^T, U^T) - C(p^T, U^0) \; ;$$

(2) $$EV = C(p^0, U^T) - C(p^0, U^0) \; .$$

Let *observed* income or expenditure $C(p^0, U^0)$ change to $C(p^T, U^T)$ and define the change as,

(3) $$\Delta I = C(p^T, U^T) - C(p^0, U^0) \; .$$

By combining (1), (2), and (3), CV and EV become,

(4) $$CV = C(P^0, U^0) - C(p^T, U^0) + \Delta I \; ;$$

(5) $$EV = C(p^0, U^T) - C(p^T, U^T) + \Delta I \; .$$

If (3) is zero, (4) and (5) conform to the original Hicksian definition.[2]

The Konüs-type true indices of the cost of living (Konüs, 1939; Pollak, 1971; Diewert, 1980) may be denoted as $I(p^0, p^T; U^0)$ and $I(p^0, p^T; U^T)$ where U^0 or U^T is the reference "standard of living" in each instance. By definition,

(6) $$I(p^0, p^T; U^0) = \frac{C(p^T, U^0)}{C(p^0, U^0)} \quad ; \quad I(p^0, p^T; U^T) = \frac{C(p^T, U^T)}{C(p^0, U^T)} \; .$$

In practice, these are approximated by the Laspeyres price index, which is the *upper* bound to the true index for U^0, and by the Paasche price index, which is the *lower* bound to the true index

[2]Following Hicks (1956), CV uses "old" utility U^0 whereas EV uses "new" utility U^T as the fixed reference utility level. While this utility referencing is universally followed, there is no universal convention on the sign of CV and EV. The formulations above are the same as in Boadway and Bruce (1984), Varian (1984), and Cornes (1992) where CV and EV are positive when utility levels rise, U^T higher than U^0 in (1) and (2). It appears that the signs of CV and EV indicate the *direction of welfare change*. Equivalently, CV and EV are positive when prices fall, p^T lower than p^0 in (4) and (5). This is because the expenditure function $C(p, U)$ is non-decreasing in utility, given the same prices, and also non-decreasing in prices, given the same utility.

Other authors, e.g., Deaton and Muellbauer (1980b) and Hausman (1981), define CV and EV with the opposite sign while they follow the same utility referencing. In this case, the signs of CV and EV indicate the *direction of compensation*. For example, CV and EV are negative if prices fall. Hence, the *negative* sign indicates that CV or EV is the amount of income that could be *taken away* from the consumer so that he remains at U^0 or U^T. If prices rise, CV are EV are *positive* so that each measures the amount of income that should be *given to* the consumer to keep him at U^0 or U^T. While CV and EV are always of the same sign, they need not be equal in value.

for U^T (Konüs, 1939; Deaton and Muellbauer, 1980b). These bounds can be established by invoking duality and expenditure minimization.[3]

From (4), (5), and (6), welfare changes and cost of living indices are measured by computing the compensated incomes $C(p^T, U^0)$ and $C(p^0, U^T)$ starting, respectively, from the observed incomes $C(p^0, U^0)$ and $C(p^T, U^T)$. While compensated incomes are unobservable in theory, they are derivable from ordinary demand functions estimated in practice from price and income data. For this reason, an algorithm for deriving compensated incomes from ordinary demand functions is valuable for rigorous analysis of the effects of price changes on welfare levels and on the costs of maintaining a standard of living.

3. The RESORT Algorithm

Let $C^0 = C(p^0, U^0)$ and $C^T = C(p^T, U^0)$. Also, let t be an auxiliary variable in the interval $0 \leq t \leq T$ such that $p(t)$ is a differentiable price curve connecting p^0 to p^T. By continuity in prices of the expenditure function, the change from C^0 to C^T can be expressed as

[3]Let q^0 be the quantity bundle bought at p^0 and q^T be the bundle bought at p^T. Define the costs by the dot products $p^0 \cdot q^0$, $p^T \cdot q^T$, $p^0 \cdot q^T$, and $p^T \cdot q^0$. The Laspeyres price index (I_p^L) and Paasche price index (I_p^P) are

$$I_p^L = \frac{p^T \cdot q^0}{p^0 \cdot q^0} \quad ; \quad I_p^P = \frac{p^T \cdot q^T}{p^0 \cdot q^T} .$$

By duality, $C(p^0, U^0) = p^0 \cdot q^0$ and $C(p^T, U^T) = p^T \cdot q^T$. Expenditure minimization implies that $C(p^0, U^T) \leq p^0 \cdot q^T$ and $C(p^T, U^0) \leq p^T \cdot q^0$. It follows that

$$I(p^0, p^T; U^0) = \frac{C(p^T, U^0)}{C(p^0, U^0)} \leq I_p^L = \frac{p^T \cdot q^0}{p^0 \cdot q^0} \quad ; \quad I(p^0, p^T; U^T) = \frac{C(p^T, U^T)}{C(p^0, U^T)} \geq I_p^P = \frac{p^T \cdot q^T}{p^0 \cdot q^T} .$$

Alternatively, given that the expenditure function is concave in prices, these inequalities can be established from the fact that the Laspeyres and Paasche price indices are first-order Taylor series approximations to their corresponding true cost of living indices.

$$(7) \qquad C^T = C^0 + \sum_{i=1}^{n} \int_0^T \frac{\partial C(p(t), U^0)}{\partial p_i(t)} \, dp_i(t) = C^0 + \sum_{i=1}^{n} \int_0^T H_i(p(t), U^0) \, dp_i(t)$$

where $H_i(p(t), U^0)$ is a compensated demand function by Shephard's lemma.

The analysis will focus on computing (7) where $C^T = C(p^T, U^0)$ is being measured starting from $C^0 = C(p^0, U^0)$ as prices change from and p^0 to p^T. In principle, the procedure applies equally to measuring $C(p^0, U^T)$ from $C(p^T, U^T)$ as prices change back from p^T to p^0. For this reason, the discussion in the rest of the paper will consider only the computation of (7) to avoid unnecessary repetition.

Following Vartia, let the total change in each price from $p^0 = \{p_i^0\}$ to $p^T = \{p_i^T\}$ be broken into price steps. Hence, suppose that there are steps

$$(8) \qquad s = 0, \ldots, Z \quad ; \quad 1 \le Z < \infty$$

where Z is a positive integer. Thus, the price at each step is

$$(9) \qquad p_i(s+1) = p_i(s) + \frac{1}{Z}(p_i^T - p_i^0) \quad ; \quad p_i(0) = p_i^0 \quad ; \quad p_i(Z) = p_i^T .$$

Starting from the original prices, p_i^0, the prices at the last step $s = Z$ are the same as the terminal prices, p_i^T.

Let q be an auxiliary variable in the interval $s \le q \le s+1$. Hence, given (9), the analogous equation to (7) is

$$(10) \qquad C(s+1) = C(s) + \sum_{i=1}^{n} \int_s^{s+1} H_i(p(q), U^0) \, dp_i(q) .$$

In (10), the starting value of compensated income is C^0. The terminal value is obtained by adding to C^0 the sum of the changes in compensated income from each step. That is,

(11) $$C^T = C^0 + \sum_{s=0}^{Z} (C(s+1) - C(s)) \quad ; \quad C(0) = C^0 .$$

Therefore, the solution for C^T is the value of $C(s)$ at the last step Z, i.e.,

(12) $$C(Z) = C^T .$$

To calculate (10), consider first the fact that it can be expressed as a Taylor series expansion around the starting value $C(s)$. That is, for an rth-order Taylor series expansion with a remainder R,

(13) $$C(s+1) = C(s) + \sum_{m=1}^{r} \frac{1}{m!} d^m C(p(q), U^0) + R$$

where $d^m C(p(q), U^0)$ is the total differential of order m of the expenditure function as the prices $p(q)$ change from $p(s)$ to $p(s+1)$. Given a non-zero remainder, the approximation could achieve arbitrary accuracy depending on the choice of the highest order r of the Taylor series in (13).

Consider approximations up to the second-order. By duality,

(14) $$\frac{\partial C(p(q), U^0)}{\partial p_i}\Big|_{q=s} = H_i(p(s), U^0) = h_i(p(s), C(s))$$

where $h_i(p(s), C(s))$ is the ordinary demand function. That is, when compensated income is substituted into the ordinary demand functions, the quantities obtained are the compensated quantities. Moreover, using general notation, the Slutsky equation yields,

(15) $$\frac{\partial^2 C(p,U)}{\partial p_i \partial p_j} = \frac{\partial H_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} + h_j \frac{\partial h_i}{\partial C} = S_{ij}$$

where S_{ij} denotes the compensated (Hicksian) price effect.

Recalling (9), the changes in prices from one step to the next are given by,

(16) $$\Delta p_i = p_i(s+1) - p_i(s) = \frac{1}{Z}(p_i^T - p_i^0) .$$

Substituting (14) to (16) into (13) and ignoring the remainder term R, the second-order Taylor series approximation $C_r(s+1)$ to the true compensated income $C(s+1)$ is,

(17) $$C_r(s+1) = C_r(s) + \sum_{i=1}^{n} h_i(p(s),C_r(s))\,\Delta p_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} S_{ij}(p(s),C_r(s))\,\Delta p_i \Delta p_j .$$

The computation of (17) starting at $s = 0$ begins with the given compensated income $C_r(0) = C(0) = C^0$ at the initial price vector $p(0) = p^0$. At any step $s+1$, the computation requires that the ordinary demand functions and its derivatives be evaluated given the known prices and the compensated income from the preceding step s. In this view, (17) is a "forward" second-order approximation.

Suppose that (17) has been computed all the way to the last step $s = Z$, where $C_r(Z)$ is the approximation to the true compensated income C^T at the terminal price vector $p(Z) = p^T$. Technically, the "forward" approximation may be reversed starting with $C_r(Z)$ and $p(Z)$. That is, $C_r(s)$ is to be solved knowing $C_r(s+1)$ as prices change from $p(s+1)$ to $p(s)$. Hence, using (16), the reverse of (17) or the "backward" second-order approximation to $C(s)$ is,

(18) $$C_r(s) = C_r(s+1) - \sum_{i=1}^{n} h_i(p(s+1),C_r(s+1))\,\Delta p_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} S_{ij}(p(s+1),C_r(s+1))\,\Delta p_i \Delta p_j .$$

The "solution" to $C_r(s)$ in (18) will not necessarily be the same as its "known" value in (17). Similarly, the "solution" to $C_r(s+1)$ in (17) will not necessarily be the same as its "known" value in (18).[4] To insure that (17) and (18) give the same values of $C_r(s)$ and $C_r(s+1)$, combine

[4]While the analytic basis may not be obvious, this claim may be verified numerically.

the two equations and solve $C_r(s+1)$ as the mutual unknown from,

$$(19)\qquad C_r(s+1) = C_r(s) + \frac{1}{2}\sum_{i=1}^{n} h_i(p(s),C_r(s))\,\Delta p_i + \frac{1}{2}\sum_{i=1}^{n} h_i(p(s+1),C_r(s+1))\,\Delta p_i$$

$$+ \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} S_{ij}\,(p(s),C_r(s))\;\Delta p_i\,\Delta p_j$$

$$- \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} S_{ij}\,(p(s+1),C_r(s+1))\;\Delta p_i\,\Delta p_j$$

starting from $C_r(0) = C(0) = C^0$. Because $C_r(s+1)$ is in both sides of (19), the solution requires iteration. In (19), the values of $C_r(s)$ and $C_r(s+1)$ must satisfy both the "forward" solution in (17) and the "backward" solution in (18). Thus, (19) is a REversible Second-ORder Taylor (RESORT) algorithm for computing compensated income.

The RESORT approximation to the true compensated income C^T in (7) is given by the solution of (19) at the last step $s = Z$, i.e.,

$$(20)\qquad C_r(Z) \approx C^T = C^0 + \sum_{i=1}^{n} \int_0^T H_i(p(t),U^0)\,dp_i(t)\ .$$

Obviously, the iterative solution of (19) can only begin forward from the initial compensated income $C(0) = C^0$ at prices $p_i(0) = p_i^0$. However, once the terminal solution $C_r(Z)$ at prices $p_i(Z) = p_i^T$ is obtained, reversibility means that RESORT can reproduce exactly each intermediate compensated income approximation $C_r(s)$ at prices $p_i(s)$, $0 \le s \le Z$, starting backwards from $C_r(Z)$. Thus, by solving simultaneously the ordinary second-order Taylor approximations in (17) and (18), the RESORT algorithm in (19) has the ideal feature of giving a unique value of compensated income for each price vector, which is not necessarily true when (17) and (18) are solved separately.

Numerical simulations using the same examples in section 5 show that (17) and RESORT in (19) give equally close approximations, up to four decimal places, to the true compensated income at each price step. Note that both (17) and (19) are step-by-step approximations from price step 0 to 1, 1 to 2 or, in general, from s to $s+1$, $s = 0, 1, \ldots, Z$. However, (17) does not have the reversibility feature of RESORT and, thus, (17) does not guarantee a unique backward and forward solution of compensated income at each price step.

The usual second-order Taylor approximation calculates (17) but always starting from the original prices, i.e., a one-step procedure starting from 0 to 1, 0 to 2, or from 0 to any price step. In this case, numerical simulations also show that (17) or RESORT in (19) implemented step-by-step give closer approximations to the true compensated income than (17) implemented in one step. Therefore, to obtain precise and unique approximations to compensated income, RESORT is the preferred second-order Taylor series procedure to compete with Vartia's numerical integration. Other than mere precision, however, RESORT has other advantages over Vartia's procedure as discussed below.

The expenditure function is concave and linearly homogeneous in prices. By rewriting the second-order terms in (19), concavity implies that (Samuelson, 1947; Sydsaeter, 1981),

$$(21) \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} S_{ij}\,\Delta p_i\,\Delta p_j = \begin{bmatrix}\Delta p_1 & \Delta p_2 & \cdots & \Delta p_n\end{bmatrix} \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \cdots & \ddots & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix} \begin{bmatrix}\Delta p_1 \\ \Delta p_2 \\ \vdots \\ \Delta p_n\end{bmatrix} \leq 0 \, .$$

Linear homogeneity in prices of the expenditure function implies

$$
(22) \qquad \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \cdots & \ddots & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} .
$$

In particular, (22) states that compensated demand functions are zero-degree homogeneous in prices.[5] In (21) or (22), the square matrix of compensated price effects, S_{ij}, is symmetric by Young's theorem because it is the Hessian of the expenditure function. This matrix is negative semi-definite because of concavity, which is equivalent to saying that it has non-positive eigenvalues. Moreover, linear homogeneity implies that it is singular, which is equivalent to saying that at least one eigenvalue is zero. Therefore, the presence of a positive eigenvalue is sufficient evidence of a violation of concavity in prices and the absence of a zero eigenvalue is sufficient evidence of a violation of linear homogeneity in prices.

In practical applications when only the ordinary demand functions are known, the RESORT algorithm computes S_{ij} through the Slustky equation in (15). This equation can be rewritten in terms of expenditure shares and ordinary price and expenditure or income elasticities,

$$
(23) \qquad S_{ij} = \frac{w_i C}{p_i p_j} \left[\frac{\partial h_i}{\partial p_j} \frac{p_j}{h_i} + \frac{\partial h_i}{\partial C} \frac{C}{h_i} w_j \right]
$$

where $h_i = h_i(p, C)$ is the ordinary demand function. The property in (22) means zero-degree homogeneity in prices of the compensated demand function, which in turn is equivalent to zero-degree homogeneity in prices and expenditure of the ordinary demand function. It follows from (22) that (23) yields,

[5]The result in (22) follows from Euler's theorem on homogeneous functions because a compensated demand function is, by Shephard's lemma, the own-price derivative of the expenditure function.

(24) $$\sum_{j=1}^{n} S_{ij} p_j = \frac{w_i C}{p_i} \left[\sum_{j=1}^{n} \frac{\partial h_i}{\partial p_j} \frac{p_j}{h_i} + \frac{\partial h_i}{\partial C} \frac{C}{h_i} \sum_{j=1}^{n} w_j \right] = 0$$

which is true because,

(25) $$\sum_{j=1}^{n} \frac{\partial h_i}{\partial p_j} \frac{p_j}{h_i} + \frac{\partial h_i}{\partial C} \frac{C}{h_i} = 0 \quad ; \quad \sum_{j=1}^{n} w_j = 1 \, .$$

This is the property of ordinary demand functions that the sum of price and income elasticities equals zero. Moreover, the sum of expenditure shares equals one from the budget constraint.

Checking that both (21) and (22) are satisfied at every price set $p_i(s)$, from the initial set p_i^0 to the terminal set p_i^T, is important because if either one is violated then the change in compensated income from one price step to the next could not have been a move from an expenditure-minimizing or utility-maximizing point. This feature is important in empirical work when it cannot be presumed that the demand system being used is globally well-behaved, or locally well-behaved over the price range under examination.

In a two-good case where each good satisfies (24) or (25) and where the compensated cross-price effects are symmetric, a necessary and sufficient condition for (21) is that the compensated own-price effect for either good is non-positive. This implies the condition from (23) that,

(26) $$S_{ii} = \frac{w_i C}{p_i^2} \left[\frac{\partial h_i}{\partial p_i} \frac{p_i}{h_i} + \frac{\partial h_i}{\partial C} \frac{C}{h_i} w_i \right] \leq 0 \quad , \quad i = 1,2 \, .$$

This condition is easily visualized in the two-good case.[6]

[6]Suppose that the utility function is strictly quasi-concave so that the indifference curves are strictly convex with respect to the origin. In this case, an interior or tangency point with a budget line is utility-maximizing (expenditure-minimizing). At a tangency point, $S_{ij} > 0$ because the two goods can only be Hicksian substitutes. This implies from (24) that $S_{ii} < 0$. If the utility function is not quasi-concave, i.e., indifference curves are not convex with respect to the origin, the utility-maximizing (expenditure-minimizing) solution is at a corner point, at the intercept of the budget line in one of the axes. At the corner, $S_{ij} = 0$ so that (24) implies $S_{ii} = 0$. Thus, given

In section 4, all the ordinary demand functions satisfy symmetry of compensated price effects and zero-degree homogeneity in prices and expenditure. However, concavity in prices of the expenditure function is not always satisfied. Thus, (26) will be used by RESORT to test for violation of concavity at every price step in the case of two goods.

4. Vartia's Approximation by Numerical Integration

Vartia originally proposed a step-by-step approximation by his construction of the price steps in (9). In this framework, he derived an expression similar to (10) by equating the total differential of an indirect utility function to zero, i.e., holding utility constant. However, in place of the compensated demand function in the right-hand side of (10) Vartia had the ordinary demand function. This is warranted because the compensated quantity of the good is obtained when compensated income is substituted into the ordinary demand function. That is,

$$(27) \qquad H_i(p(q), U^0)\, dp_i(q) \;=\; h_i(p(q), C(q))\, dp_i(q)$$

since $C(q)$ is compensated income that maintains utility at U^0. In view of (27), Vartia's algorithm to compute (10) is based on the following approximation by numerical integration of an ordinary demand function,

$$(28) \qquad \int_s^{s+1} h_i(p(q), C(q))\, dp_i(q) \;\approx\; \frac{1}{2}\left[h_i(p(s+1), C(s+1)) \;+\; h_i(p(s), C(s))\right](p_i(s+1) - p_i(s)) \;.$$

---

only two goods, $S_{ii} \leq 0$ in (26) is necessary and sufficient for a negative semi-definite matrix of compensated price effects (expenditure minimization).

Incidentally, this two-good case illustrates that concavity in prices (negative semi-definite matrix above) of the expenditure function does not imply quasi-concavity of the utility function (Deaton and Muellbauer, 1980b; Cornes, 1992). Expenditure minimization is possible at the corner when the utility function is not quasi-concave. Quasi-concavity of the utility function, however, implies expenditure minimization and, therefore, concavity in prices of the expenditure function.

Combining (10), (27), and (28) and replacing $C(s+1)$ by the approximation value $C_v(s+1)$,

$$(29) \qquad C_v(s+1) = C_v(s) + \sum_{i=1}^{n} \frac{1}{2}\left[h_i(p(s+1), C_v(s+1)) + h_i(p(s), C_v(s))\right](p_i(s+1) - p_i(s))$$

is Vartia's algorithm. This is implemented by following the same price steps in (9) and starting with the initial value of compensated income. At each step,

$$(30) \qquad C_v(s) \approx C(s)\ ,\ 0 \le s \le Z \quad ; \quad C_v(0) = C(0) = C^0 \quad ; \quad C_v(Z) \approx C^T .$$

From (29) and (30), Vartia's approximation to the true compensated income C^T at the last step Z is $C_v(Z)$. Except for differences in notation, (29) is an exact representation of Vartia's algorithm because it can reproduce all the results in his numerical illustrations.

Like RESORT in (19), Vartia's algorithm in (29) is also reversible. Having obtained $C_v(Z)$ and using this as a starting value, (29) can reproduce $C_v(s)$ exactly at each step $0 \le s \le Z$. In contrast to RESORT, however, Vartia's procedure does not require knowledge about the matrix of compensated price effects. Thus, while RESORT could catch demand systems that violate the theoretical restrictions, Vartia's procedure would let them slip undetected.

5. Numerical Examples

Example 1: McKenzie-Pearce Utility Function

This example reproduces exactly the compensated incomes computed in the original application of Vartia's algorithm. These were obtained from the ordinary demand functions derived from the indirect utility function $U(p, C)$ of McKenzie and Pearce (1976) given by

$$(31)\qquad U(p,C) = \frac{C}{p_1} + \frac{C}{p_2} \quad ; \quad C(p,U) = \frac{p_1 p_2 U}{p_1 + p_2}$$

where $C(p, U)$ is the dual expenditure function. The goods are x_1 and x_2 with prices p_1 and p_2. By Roy's identity, (31) yields the ordinary demand functions,

$$(32)\qquad h_1(p,C) = \frac{p_2 C}{p_1(p_1 + p_2)} \quad ; \quad h_2(p,C) = \frac{p_1 C}{p_2(p_1 + p_2)}$$

which were also derived by Vartia, although these were reported incorrectly elsewhere in his paper.[7] The data are as follows:

$$(33)\qquad \{p_1^0, p_2^0\} = \{1, 2\} \quad ; \quad \{p_1^T, p_2^T\} = \{1.2375, 1.2692\} \quad ; \quad U^0 = V(p^0, C^0) = 330 \ ;$$

$$(34)\qquad \{h_1^0, h_2^0\} = \{146.6667, 36.6667\} \quad ; \quad C^0 = p_1^0 h_1^0 + p_2^0 h_2^0 = 220 \ .$$

In Vartia's Table III in his appendix, there are only eight price steps or $Z = 8$. However, with only 8 steps, the illustration is not too illuminating because the difference between the initial and terminal levels of compensated income turned out to be very small. Hence, this paper modified Vartia's terminal price vector by increasing the price of good 1 further to 1.2375 and decreasing the price of good 2 further to 1.2692. However, the changes in the prices from one step to the next are the same as in Vartia's original example. The result is that the number of price steps is now up to Z = 19. Therefore, using (9), the prices are obtained from,

$$(35)\qquad p_1(s+1) = p_1(s) + \frac{1}{Z}(p_1^T - p_1^0) = p_1(s) + 0.0125 \ ;$$

$$(36)\qquad p_2(s+1) = p_2(s) + \frac{1}{Z}(p_2^T - p_2^0) = p_2(s) - 0.0384625 \ .$$

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[7]These ordinary demand functions are correct and are the same as those reported by Vartia in his Appendix 3, page 95. Unfortunately, by typographical error, Vartia incorrectly reported these demand functions as $(p_2/p_1)/(C/(p_1+p_2)$ for h_1 and $(p_1/p_2)/(C/(p_1+p_2)$ for h_2 on page 96.

By substituting the prices from (35) and (36) and the demand functions in (32) into Vartia's algorithm in (29), $C_v(s)$ can be solved for each price pair, $\{p_1(s), p_2(s)\}$, at each step starting from the original value $C_v(0) = C^0$. The values reported by Vartia in his Table III are the same as the values reported in Table 1 below at each step starting from the initial situation up to step 8.

It can be verified that the ordinary demand functions in (32) are homogeneous of degree zero in prices and income. Also, the expenditure function $C(p, U)$ in (31) is linearly homogeneous in prices and the compensated price effects are symmetric. With only two goods, a non-positive compensated own-price effect for one of the goods is necessary and sufficient for the concavity in prices of the expenditure function. This condition is satisfied for the range of prices in Table 1, as shown by the negative compensated own-price effect for good 1.[8]

Table 1 shows the true compensated income at each price step obtained from the expenditure function in (31). Vartia's computations of compensated income are closer to the true compensated income than the computations from RESORT. However, Table 1 shows that the results are almost always the same up to two decimal places. In any case, the issue of precision appears to be an empirical matter, depending on specific cases. This is illustrated by the AIDS model in Table 2 where RESORT gives closer approximations to the true compensated income at each price step than Vartia's algorithm.

[8]Compensated price effects are the second-order price derivatives of the expenditure function. If the expenditure function is unknown, compensated price effects could be obtained by deriving the price and income elasticities from the ordinary demand functions and then substituting these elasticities, together with the expenditure shares, into the Slutsky equations in (23) and (26) for cross-price and own-price effects, respectively.

Table 1

Compensated Income From the McKenzie-Pearce Utility Function

Price Steps	Price of Good 1	Price of Good 2	Compensated Income			Compensated Own-Price Effect of Good 1
			True Value	Vartia's Approximation	RESORT Approximation	
0	1.0000	2.0000	220.0000	220.0000	220.0000	- 97.7778
1	1.0125	1.9615	220.3734	220.3732	220.3738	- 96.5379
2	1.0250	1.9231	220.6457	220.6453	220.6466	- 95.2622
3	1.0375	1.8846	220.8143	220.8136	220.8156	- 93.9501
4	1.0500	1.8461	220.8763	220.8754	220.8782	- 92.6008
5	1.0625	1.8077	220.8289	220.8278	220.8313	- 91.2137
6	1.0750	1.7692	220.6691	220.6677	220.6719	- 89.7880
7	1.0875	1.7308	220.3937	220.3920	220.3971	- 88.3232
8	1.1000	1.6923	219.9996	219.9976	220.0035	- 86.8185
9	1.1125	1.6538	219.4834	219.4811	219.4879	- 85.2735
10	1.1250	1.6154	218.8416	218.8390	218.8467	- 83.6875
11	1.1375	1.5769	218.0706	218.0677	218.0763	- 82.0600
12	1.1500	1.5384	217.1667	217.1635	217.1730	- 80.3905
13	1.1625	1.5000	216.1260	216.1225	216.1329	- 78.6786
14	1.1750	1.4615	214.9444	214.9406	214.9520	- 76.9238
15	1.1875	1.4231	213.6178	213.6137	213.6261	- 75.1259
16	1.2000	1.3846	212.1418	212.1373	212.1507	- 73.2846
17	1.2125	1.3461	210.5117	210.5069	210.5215	- 71.3997
18	1.2250	1.3077	208.7230	208.7178	208.7335	- 69.4713
19	1.2375	1.2692	206.7707	206.7651	206.7819	- 67.4993

As noted before, numerical simulations show that RESORT in (19) gives equally close approximations to the true compensated income as the second-order approximation in (17) implemented as step-by-step procedure like (19), from 0 to 1, 1 to 2, ..., 17 to 18 and 18 to 19. However, RESORT gives much closer approximations compared to (17) implemented as a one-step procedure, i.e., always starting (17) from 0 to any price step. For these reasons, only the results from RESORT are reported in the tables.

Example 2: A Well-behaved Almost Ideal Demand System (AIDS)

Consider the indirect utility function,

$$(37) \qquad U(p,C) = \left(\frac{C}{A}\right)^{\frac{1}{B}}$$

where p is the price vector and C is income. A and B are price functions defined by

$$(38) \qquad \ln A = \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln p_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij} \ln p_i \ln p_j \, ;$$

$$(39) \qquad \ln B = \beta_0 + \sum_{i=1}^{n} \beta_i \ln p_i \, .$$

Combining (37), (38), and (39) and then using Roy's identity, the indirect utility function yields the ordinary demand function $h_i(p, C)$ for good i,

$$(40) \qquad h_i(p,C) = \frac{C}{p_i}\left[\alpha_i + \sum_{j=1}^{n} \gamma_{ij} \ln p_j + \beta_i \ln\left(\frac{C}{A}\right)\right].$$

By definition, an expenditure share is,

(41) $$w_i = \frac{p_i h_i(p,C)}{C} .$$

Thus, (40) and (41) yield,

(42) $$w_i = \alpha_i + \sum_{j=1}^{n} \gamma_{ij} \ln p_j + \beta_i \ln\left(\frac{C}{A}\right)$$

which is the expenditure share function of the AIDS model (Deaton and Muellbauer, 1980a, 1980b). The AIDS parameter restrictions are,

(43) $$\sum_{i=1}^{n} \alpha_i = 1 \ ; \ \sum_{i=1}^{n} \beta_i = 0 \ ; \ \sum_{i=1}^{n} \gamma_{ij} = 0 \ ; \ \sum_{j=1}^{n} \gamma_{ij} = 0 \ ; \ \gamma_{ij} = \gamma_{ji} .$$

These restrictions insure additivity of shares to one; zero-degree homogeneity in income and prices of the ordinary demand functions; and symmetry of the compensated price effects.

Consider the case of two goods. The AIDS ordinary demand function in (40) yields the elasticities,

(44) $$E_{ii} = \frac{\partial h_i}{\partial p_i}\frac{p_i}{h_i} = -1 + \frac{1}{w_i}\left\{\gamma_{ii} - \beta_i\left[w_i - \beta_i \ln\left(\frac{C}{A}\right)\right]\right\} ;$$

(45) $$E_{ij} = \frac{\partial h_i}{\partial p_j}\frac{p_j}{h_i} = \frac{1}{w_i}\left\{\gamma_{ij} - \beta_i\left[w_j - \beta_j \ln\left(\frac{C}{A}\right)\right]\right\} ;$$

(46) $$E_{iC} = \frac{\partial h_i}{\partial C}\frac{C}{h_i} = 1 + \frac{\beta_i}{w_i} .$$

By substituting (44) and (46) into (26), the compensated own-price effect for either good is

(47) $$S_{ii} = -\frac{C}{p_i^2}\left[(1 - w_i) w_i - \gamma_{ii} - \beta_i^2 \ln\left(\frac{C}{A}\right)\right] .$$

In Table 2, the following parameter values are used,

$$\text{(48)} \qquad \alpha_0 = 0 \ ; \ \beta_0 = 0 \ ; \ \alpha_1 = -0.45 \ ; \ \beta_1 = 0.10 \ ; \ \gamma_{11} = -0.20 \ .$$

The values of the other parameters, α_2, β_2, γ_{12}, γ_{21}, and γ_{22}, are obtained from (48) subject to the parameter restrictions in (43). The price data for the initial and terminal situations, as well as the price steps, are the same as in example 1. However, to yield the same initial income, the fixed level of utility is set equal to 100. That is,

$$\text{(49)} \qquad \{p_1^0, p_2^0\} = \{1, 2\} \ ; \ \{p_1^T, p_2^T\} = \{1.2375, 1.2692\} \ ; \ C^0 = 220 \ ; \ U^0 = 100 \ .$$

The true compensated income in this AIDS model can be obtained by solving for $C(p, U)$ from (37) and using the parameters in (48) and (49). The compensated income approximations at each price step using RESORT in (19) and Vartia's algorithm in (29) are shown in Table 2. Moreover, notice that the above two-good AIDS model is well-behaved in the price range under examination from the fact that the compensated own-price effect for good 1 is negative.

In Table 2, the RESORT approximation to the true compensated income is closer than Vartia's approximation at each price step. Thus, given Vartia's closer approximations in Table 1, the relative precision of one procedure over the other is case specific. Therefore, these results show that Vartia's algorithm or RESORT is not necessarily superior over the other simply in terms of the closeness of the approximations to the true compensated income.

Table 2

Compensated Income From a Well-behaved AIDS Model

Price Steps	Price of Good 1	Price of Good 2	Compensated Income			Compensated Own-Price Effect of Good 1
			True Value	Vartia's Approximation	RESORT Approximation	
0	1.0000	2.0000	220.0000	220.0000	220.0000	- 61.7870
1	1.0125	1.9615	216.7793	216.7604	216.7792	- 58.6254
2	1.0250	1.9231	213.4973	213.4787	213.4970	- 55.5901
3	1.0375	1.8846	210.1547	210.1364	210.1543	- 52.6759
4	1.0500	1.8461	206.7524	206.7344	206.7519	- 49.8776
5	1.0625	1.8077	203.2912	203.2735	203.2905	- 47.1902
6	1.0750	1.7692	199.7718	199.7544	199.7710	- 44.6094
7	1.0875	1.7308	196.1949	196.1779	196.1941	- 42.1306
8	1.1000	1.6923	192.5615	192.5447	192.5605	- 39.7499
9	1.1125	1.6538	188.8721	188.8557	188.8710	- 37.4634
10	1.1250	1.6154	185.1276	185.1115	185.1264	- 35.2674
11	1.1375	1.5769	181.3287	181.3129	181.3274	- 33.1585
12	1.1500	1.5384	177.4760	177.4606	177.4746	- 31.1335
13	1.1625	1.5000	173.5704	173.5553	173.5689	- 29.1893
14	1.1750	1.4615	169.6125	169.5978	169.6109	- 27.3230
15	1.1875	1.4231	165.6031	165.5887	165.6014	- 25.5318
16	1.2000	1.3846	161.5429	161.5289	161.5411	- 23.8132
17	1.2125	1.3461	157.4327	157.4191	157.4308	- 22.1648
18	1.2250	1.3077	153.2732	153.2600	153.2713	- 20.5842
19	1.2375	1.2692	149.0653	149.0524	149.0633	- 19.0692

Example 3: A Mis-behaved AIDS Model

The next example is designed to misbehave in order to show that Vartia's procedure could yield compensated incomes that unknowingly are theoretically invalid. Keeping everything else the same as in example 2, let the parameters in (48) be changed to,

(50) $\alpha_0 = 0 \ ; \ \beta_0 = 0 \ ; \ \alpha_1 = -0.8882 \ ; \ \beta_1 = 0.30 \ ; \ \gamma_{11} = -0.25$.

The values of α_2, β_2, γ_{12}, γ_{21}, and γ_{22} are in this case derived from (50) subject to the parameter restrictions in (43). In this case, the two-good AIDS model is misbehaved from the fact that the compensated own-price effect for good 1 turns from negative to positive as shown in Table 3.

It is interesting to note in Table 3 that the RESORT approximations to the true compensated income are closer than Vartia's approximations for the range of prices with negative compensated own-price effects. In this range, the compensated incomes are theoretically valid. However, Vartia's approximations are closer for the range of prices with positive compensated own-price effects when compensated incomes are theoretically invalid. This case illustrates the advantage of RESORT over Vartia's algorithm in catching violations of the theoretical restrictions that would have been undetected by Vartia's procedure.

The next example utilizes ordinary demand functions from a model without an explicit utility function. This is precisely the situation for which Vartia's algorithm and RESORT are intended because the values of the true compensated income are not exactly known and can only be approximated from the ordinary demand functions.

Table 3

Compensated Income From a Mis-behaved AIDS Model

Price Steps	Price of Good 1	Price of Good 2	Compensated Income			Compensated Own-Price Effect of Good 1
			True Value	Vartia's Approximation	RESORT Approximation	
0	1.0000	2.0000	220.0000	220.0000	220.0000	- 12.7086
1	1.0125	1.9615	220.4166	220.4143	220.4164	- 11.1791
2	1.0250	1.9231	220.8215	220.8193	220.8212	- 9.6764
3	1.0375	1.8846	221.2159	221.2138	221.2154	- 8.1974
4	1.0500	1.8461	221.6010	221.5990	221.6002	- 6.7393
5	1.0625	1.8077	221.9782	221.9762	221.9772	- 5.2990
6	1.0750	1.7692	222.3488	222.3470	222.3475	- 3.8734
7	1.0875	1.7308	222.7145	222.7128	222.7128	- 2.4596
8	1.1000	1.6923	223.0768	223.0752	223.0749	- 1.0544
9	1.1125	1.6538	223.4377	223.4363	223.4354	0.3453
10	1.1250	1.6154	223.7992	223.7979	223.7964	1.7430
11	1.1375	1.5769	224.1633	224.1622	224.1601	3.1420
12	1.1500	1.5384	224.5325	224.5316	224.5289	4.5460
13	1.1625	1.5000	224.9095	224.9089	224.9054	5.9588
14	1.1750	1.4615	225.2972	225.2968	225.2924	7.3846
15	1.1875	1.4231	225.6989	225.6987	225.6934	8.8278
16	1.2000	1.3846	226.1181	226.1183	226.1119	10.2932
17	1.2125	1.3461	226.5590	226.5596	226.5519	11.7860
18	1.2250	1.3077	227.0261	227.0272	227.0181	13.3120
19	1.2375	1.2692	227.5247	227.5263	227.5156	14.8776

Example 4: A Generalized Logit Demand System

Approximations to compensated income are, in principle, unnecessary in the preceding three examples because the true compensated incomes are known from the utility function or expenditure function. These three examples are, however, useful for gauging the relative accuracy of the compensated income approximations from Vartia's algorithm and RESORT by comparing their results to the true values of compensated income. This comparison is not possible if the ordinary demand functions did not have an explicit utility function.

Requiring that demand functions have an explicit utility function is, however, very limiting in practice. More importantly, this requirement is unnecessary in principle. For consistency with theory, it is necessary and sufficient that the demand system have a symmetric and negative semi-definite matrix of compensated price effects. If so, the demand system is "integrable," i.e., a utility function exists that could rationalize the demand system, although the utility function may not be recoverable. For this type of demand system, Vartia's algorithm and RESORT are essential for approximating the unknown true compensated income.

An example of an integrable demand system that has been successfully estimated in practice is the generalized logit demand model (Dumagan and Mount, 1993 and 1995). No utility function is posited. However, the properties implied by utility maximization or expenditure minimization are embodied into the specification, as described below.

Since expenditure shares must sum to unity, let w_i follow a generalized logit specification,

$$(51)\qquad w_i = \frac{e^{f_i}}{e^{f_1} + \ldots + e^{f_n}} = \frac{e^{f_i}}{\sum_{i=1}^{n} e^{f_i}} \quad ; \quad \sum_{i=1}^{n} w_i = 1 \ .$$

From (41) and (51), the ordinary demand function $h_i(p, C)$ is

$$(52) \qquad h_i(p,C) = \frac{C}{p_i} \frac{e^{f_i}}{\sum_{i=1}^{n} e^{f_i}} .$$

The demand system comprising $h_i(p, C)$ for all n goods can be made well-behaved, i.e., satisfy zero-degree homogeneity in p and C as well as symmetry and negative semi-definiteness of the matrix of compensated price effects, by embodying these properties in the function f_i.

The variants of the generalized logit implemented so far differ in their specifications of f_i. The following specification was implemented by Rothman, Hong, and Mount (1994),

$$(53) \qquad f_i = \alpha_i + \sum_{k=1}^{n} \delta_{ik} \theta_{ik} \ln\left(\frac{p_k}{p_i}\right) + \beta_i \ln\left(\frac{C}{SPI}\right) \quad ; \quad \delta_{ik} = \delta_{ki} \quad , \quad i \neq k$$

where α, β, and δ are parameters. SPI is a Stone price index,

$$(54) \qquad \ln SPI = \sum_{i=1}^{n} w_i \ln p_i .$$

The cross-price "weights" θ_{ik} are defined by

$$(55) \qquad \theta_{ik} = w_i^{\gamma-1} w_k^{\gamma} \quad ; \quad w_i \theta_{ik} = w_k \theta_{ki}$$

where γ is a parameter. These weights are built into the model, together with the symmetry restriction ($\delta_{ik} = \delta_{ki}$), to ensure the symmetry of the compensated price effects.

To facilitate deriving the demand elasticities of the generalized logit model, note from the share definition in (41) that the ordinary price and income elasticities can be written in terms of the share elasticities as follows:

(56) $$E_{ii} = \frac{\partial h_i}{\partial p_i}\frac{p_i}{h_i} = \frac{\partial w_i}{\partial p_i}\frac{p_i}{w_i} - 1 \; ;$$

(57) $$E_{ik} = \frac{\partial h_i}{\partial p_k}\frac{p_k}{h_i} = \frac{\partial w_i}{\partial p_k}\frac{p_k}{w_i} \quad , \quad i \neq k \; ;$$

(58) $$E_{iC} = \frac{\partial h_i}{\partial C}\frac{C}{h_i} = \frac{\partial w_i}{\partial C}\frac{C}{w_i} + 1 \; .$$

Let X be a vector of variables determining f_i, i.e., $f_i = f_i(X)$ and let x be a specific element of X. Thus, x could be price (p_i or p_k) or income (C). Therefore, from (51), the elasticity of w_i with respect to x is,

(59) $$\frac{\partial w_i}{\partial x}\frac{x}{w_i} = x\left(\frac{\partial f_i}{\partial x} - \sum_{j=1}^{n} w_j \frac{\partial f_j}{\partial x}\right) \quad , \quad \forall j \; .$$

Combining (51) to (55) and then using (56) to (59), the ordinary demand elasticities of the above generalized logit model with n goods are,[9]

(60) $$E_{ii} = -\sum_{k=1}^{n} \delta_{ik}\theta_{ik} - w_i\left(\beta_i - \sum_{j=1}^{n} w_j \beta_j\right) - 1 \; ;$$

(61) $$E_{ik} = \delta_{ik}\theta_{ik} - w_k\left(\beta_i - \sum_{j=1}^{n} w_j \beta_j\right) \; ;$$

[9]In (54) and (55), the expenditure shares are taken as "fixed" when the elasticities are derived. This means, in effect, that the price and income elasticities in (60), (61), and (62), as well as the compensated price effects in (64) and (65), are for the "short-run" when expenditure shares may be taken as "fixed." It should be noted, however, that in calculating the results in Table 4 the expenditure shares are allowed to change from one price step to the next. This should be obvious from the iterative solution of the RESORT algorithm in (19) and of Vartia's algorithm in (29). Each algorithm yields a compensated income solution at each price step and, therefore, yields the compensated quantities by substitution of this solution into the ordinary demand functions. Therefore, at each price step, there is a unique set of expenditure shares. The unique set of expenditure shares solved by RESORT at each price step is used to calculate the compensated price effects from the ordinary price and income elasticities. This procedure is used to calculate, for example, the generalized logit compensated own-price effects for good 1 reported in Table 4, based on equation (66).

(62) $$E_{iC} = \beta_i - \sum_{j=1}^{n} w_j \beta_j + 1 \ .$$

It can be verified that

(63) $$E_{ii} + \sum_{k=1}^{n} E_{ik} + E_{iC} = 0 \quad , \quad i \neq k$$

which imply that the ordinary demand functions in this generalized logit model satisfy zero-degree homogeneity in prices and income. Substituting the elasticities in (60) to (62) into the Slutsky equation in (23), it can be verified that

(64) $$S_{ik} = \frac{w_i C}{p_i p_k} \left(\delta_{ik} \theta_{ik} + w_k \right) \quad ; \quad S_{ik} = S_{ki} \ .$$

That is, the generalized logit satisfies the symmetry of the compensated cross-price effects given the symmetry of the price parameters in (53) and of the cross-price weights in (55). Moreover, the price and income elasticities and the Slutsky equation yields the compensated own-price effect,

(65) $$S_{ii} = - \frac{w_i C}{p_i^2} \left(\sum_{k=1}^{n} \delta_{ik} \theta_{ik} + 1 - w_i \right) .$$

Consider now the case with only two goods. Given that the generalized logit satisfies zero-degree homogeneity in (63) and symmetry in (64), the two-good model is well-behaved if and only if (65) is non-positive for any of the two goods, i.e., for good 1,

(66) $$S_{11} = - \frac{w_1 C}{p_1^2} \left(\delta_{12} \theta_{12} + 1 - w_1 \right) \leq 0 \ .$$

Notice from (51) that the generalized logit guarantees that each expenditure share lies strictly between 0 and 1. Hence, the cross-price weight (θ_{ik}) is positive. Therefore, since income or expenditure and prices are positive, the sign of (66) depends only on the sign of δ_{12}.

Given a positive δ_{12}, (66) is strictly negative for all prices so that the generalized logit is "integrable," implying that an underlying utility function exists in principle, although it is unknown. Therefore, an unknown underlying expenditure function also exists. Thus, except for the initial income C^0 that by duality is compensated income, the generalized logit demand model has an unknown true compensated income. In this case, the relative precision between the approximations from Vartia's method and RESORT cannot be assessed because there is no true compensated income as a basis for comparison.

For the results in Table 4, the parameters of the generalized logit model are chosen such that the initial income is the same as before. These parameters are,

$$(67) \quad \alpha_1 = 0 \;;\; \beta_1 = 0.5 \;;\; \alpha_2 = 0 \;;\; \beta_2 = 0$$
$$\delta_{12} = 1.5 \;;\; \gamma = 2.0 \;;\; C^0 = 220 \;.$$

The two approximations to compensated income are very close to each other at each price step. However, because the true compensated income is unknown, there is no basis to say that one approximation is more precise than the other.

Finally, note that this two-good generalized logit model is well-behaved from the fact that the compensated own-price effect for good 1 is negative. Indeed, (66) and (67) imply that this model is "globally" well-behaved for all positive or non-zero sets of prices and expenditures.

Table 4

Compensated Income From a Well-behaved Generalized Logit Model

Price Steps	Price of Good 1	Price of Good 2	Compensated Income			Compensated Own-Price Effect of Good 1
			True Value	Vartia's Approximation	RESORT Approximation	
0	1.0000	2.0000	220.0000	220.0000	220.0000	- 67.2522
1	1.0125	1.9615	Unknown	220.3899	220.3902	- 66.0149
2	1.0250	1.9231	Unknown	220.7128	220.7135	- 61.8006
3	1.0375	1.8846	Unknown	220.9672	220.9683	- 63.6080
4	1.0500	1.8461	Unknown	221.1517	221.1532	- 62.4360
5	1.0625	1.8077	Unknown	221.2647	221.2666	- 61.2834
6	1.0750	1.7692	Unknown	221.3043	221.3067	- 60.1492
7	1.0875	1.7308	Unknown	221.2690	221.2718	- 59.0323
8	1.1000	1.6923	Unknown	221.1566	221.1600	- 57.9317
9	1.1125	1.6538	Unknown	220.9652	220.9691	- 56.8463
10	1.1250	1.6154	Unknown	220.6927	220.6972	- 55.7752
11	1.1375	1.5769	Unknown	220.3367	220.3418	- 54.7174
12	1.1500	1.5384	Unknown	219.8948	219.9006	- 53.6719
13	1.1625	1.5000	Unknown	219.3644	219.3709	- 52.6378
14	1.1750	1.4615	Unknown	218.7428	218.7500	- 51.6141
15	1.1875	1.4231	Unknown	218.0269	218.0349	- 50.5999
16	1.2000	1.3846	Unknown	217.2136	217.2225	- 49.5942
17	1.2125	1.3461	Unknown	216.2996	216.3094	- 48.5961
18	1.2250	1.3077	Unknown	215.2812	215.2920	- 47.6046
19	1.2375	1.2692	Unknown	214.1545	214.1663	- 46.6188

6. CONCLUSION

Vartia's algorithm or the RESORT procedure in this paper permits the computation of compensated income from ordinary demand functions without having to know the underlying utility function. As a result, the true measures of welfare changes and cost of living indices are obtainable from observed price and expenditure data. The methods provided by Vartia and RESORT are important for both theoretical and practical reasons. First, the choice of demand systems in applied work need not be restricted to families having a utility function or expenditure function of an *explicit parametric form* such as the translog (Christensen, Jorgenson, and Lau, 1975; Christensen and Caves, 1980) and the "almost ideal demand system" or AIDS (Deaton and Muellbauer, 1980a and 1980b). Second, it is not necessary to recover the underlying expenditure function as suggested by Hausman (1981) to derive exact welfare measures in the case of a single price change. In any case, Hausman's method may not be practicable when more than one price changes at the same time. LaFrance (1986) and LaFrance and Hanemann (1989) found that the integrability restrictions on demand functions to obtain closed form solutions are "probably unpalatable for most applied situations," (LaFrance, 1985). Moreover, there are well-behaved demand systems in closed form for which it is impossible to recover the expenditure function in closed form (McKenzie and Ulph, 1986). Third, it offers a better alternative to approximations of Hicksian welfare measures based on the Marshallian consumer's surplus as suggested by Willig (1976) and Shonkwiler (1991). In principle, no recourse to consumer's surplus is necessary as a basis to measure the true Hicksian welfare change. In any case, consumer's surplus is of limited validity to the special cases of a single price change and of multiple price changes when the goods under study have equal income elasticities or when all income elasticities are unitary under homothetic preferences (Chipman and Moore, 1976 and 1980; Silberberg, 1972 and 1978; Just, Hueth and Schmitz, 1982; McKenzie, 1983).

A "money metric" measure of the equivalent variation from ordinary demand functions has been derived through a Taylor series expansion of the indirect utility function (McKenzie and Pearce, 1976; McKenzie, 1983). This measure is made operational by a transformation such that the marginal utility of income is stationary with a value of unity at the original income and prices. This is equivalent to saying that the marginal value of utility is also stationary and unitary so that a unit of "utility" is exactly a unit of "money" and a change in the value of the indirect utility function is itself a "money metric" measure of the change in utility. However, a money measure of welfare change need not in principle contend with the marginal utility of income. Approximations to Hicksian welfare change obtained from ordinary demand functions through a Taylor series expansion of an expenditure function do not involve the marginal utility of income and have been shown to yield more precise measures of the equivalent variation than the money metric (Dumagan and Mount, 1991).

Vartia demonstrated his algorithm using the demand functions derived from the original utility function utilized by McKenzie and Pearce to implement their money metric. Vartia showed that his procedure yields a more precise measure of compensated income and, hence, a more precise welfare measure than the money metric. Even so, this paper proposes RESORT as an alternative to Vartia's algorithm. The reason is not simply because this alternative may produce more precise approximations to compensated income but chiefly because it provides a framework to assess the validity of the results. Indeed, the vital lesson from the numerical illustrations in this paper is that the superiority of one approximation over the other cannot be based on mere precision. Either method could be more precise than the other on a case by case basis. For example, while Vartia's algorithm appears to be more precise than RESORT in Table 1, RESORT is shown to be more precise than Vartia's algorithm in Table 2 and in Table 3, for the theoretically valid values of compensated income. While obviously very important, precision is, therefore, an empirical issue that cannot be an exclusive basis for choosing one approximation method over the other.

However, the ability to check for the theoretical validity of the compensated income computations is a valuable feature of RESORT that is absent in Vartia's framework. This ability is a valid reason to choose RESORT over Vartia's algorithm because, in applications, ordinary demand functions do not always satisfy the properties implied by utility maximization or expenditure minimization.[10] That is, it is not safe to assume that the demand functions used in the approximations are well-behaved over the price range under examination. RESORT has the built-in capability to check for the theoretical validity of the computed compensated income because it calculates the elements of the matrix of compensated price effects as an integral part of the computation of compensated income. At each price step, this matrix should be evaluated for symmetry and negative semi-definiteness.

Another attractive feature of RESORT is reversibility, which is shared by Vartia's algorithm. If the computed terminal value of compensated income is used as the starting income and the price changes are reversed from the terminal levels to the initial levels, the computed income

[10]Porter-Hudak and Hayes (1991) demonstrated the practicability of Vartia's algorithm by deriving compensated incomes and cost-of-living indices from estimated ordinary demand functions, including "non-integrable" demand systems. The results from the latter systems are, however, invalid for violating the symmetry and negative semi-definiteness of the matrix of compensated price effects.

at the initial prices will be identical to the initial level of income. In fact, the compensated income at each price step is the same in the forward and backward procedures. That is, RESORT guarantees unique values of compensated income for each set of prices and, as a result, also unique measures of welfare changes and cost of living indices. These unique results are not, however, guaranteed by the usual Taylor series expansion for computing compensated income.

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