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Global Properties of Well-Behaved Demand Systems: A Generalized Logit Model Specification

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Abstract

The models analyzed in this paper satisfy additivity, homogeneity and symmetry globally *a priori* but not global negative semi-definiteness of the Hicks-Slutsky substitution matrix. Imposing parameter constraints to insure global negative semi-definiteness in a two-good model limits the range of values of the Marshallian elasticities with respect to own-price (E_{ii}), cross-price (E_{ij}) and expenditure or income (E_{iI}).

The results for the **Generalized Logit** model developed in this paper are that E_{ii} can be greater than, less than or equal to -1; E_{ij} can be greater than, less than or equal to 0; and E_{iI} can be greater than, less than or equal to 1. The results for the **Translog** and **AIDS** are qualitatively identical, namely, E_{ii} are less than -1; E_{ij} are greater than 0; and E_{iI} are equal to 1 (homothetic preferences). The results for the **Generalized Leontief** and **Minflex Laurent** are also qualitatively identical, namely, E_{ii} are equal to the negative of own-shares (negative but greater than -1); E_{ij} are equal to the negative of the share of the other good (negative but greater than -1); however, E_{iI} are also equal to 1 (homothetic preferences and zero Hicksian substitutability).

Thus, the above standard flexible functional forms can be globally well-behaved but at the expense of losing their flexibility property since they are not anymore capable of achieving the full range of price and income elasticities. The **Generalized Logit** model is also a flexible functional form and can be globally well-behaved while retaining its flexibility property from the fact that it has a wider range of price and income elasticities, therefore, encompassing more realistic cases.

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I. Background, Purpose and Summary of Findings

There are two general approaches to the modeling of empirical demand systems. One is by direct specification and the other is by deriving the demand system from a hypothesized expenditure function or indirect utility function. An example of the first approach is the logit model of expenditure shares presented in section II of this paper, which is a generalized version of the earlier logit models by Tyrrell and Mount (1982), Considine and Mount (1984) and by Considine (1990). Examples of the second approach are the flexible functional form demand models such as the "almost ideal demand system" or AIDS (Deaton and Muellbauer, 1980); the translog (Christensen, Jorgenson and Lau, 1971, 1975); the generalized Leontief (Diewert, 1971); and the minflex Laurent demand system (Barnett, 1982; Barnett and Lee, 1985).[1]

The earlier logit models satisfy additivity and zero-degree homogeneity globally for all non-negative expenditure shares. But symmetry is satisfied only at the point of means, except Considine's last specification that has global symmetry conditional on predicted shares. In contrast, the generalized logit model presented in this paper satisfies additivity, homogeneity and symmetry globally and encompasses Considine's globally symmetric model as a special case. Thus, this generalized logit model has the same features as the standard flexible form demand systems in that global additivity, homogeneity and symmetry of the Hicks–Slutsky substitution matrix (HSSM) are satisfied *a priori*. However, the restriction that the HSSM is globally negative semi-definite (GNSD) remains problematic for demand systems derived from either of these two approaches (Christensen and Caves, 1980; Barnett and Lee,

[1]Other examples of the second approach are the generalized Cobb-Douglas and generalized square root quadratic (Diewert, 1973, 1974) and the generalized Box-Cox (Berndt and Khaled, 1979).

1985; Diewert and Wales, 1987). It appears necessary to impose parameter restrictions in order for HSSM to be GNSD in additon to those required for the other global properties.[2]

For purposes of comparison, section III examines the constraints on parameters for HSSM to be GNSD in the generalized logit model, translog, AIDS, generalized Leontief and, finally, the minflex Laurent demand models. The analysis is based on the case of two goods in order to demonstrate the unrealistic implications on the price and income elasticities of the standard models of analytically constraining parameters in order for HSSM to be GNSD. This analysis also provides an insight into the difficulties of imposing negative semi-definiteness in the more general case of more than two goods.

Given that all the models analyzed satisfy additivity, homogeneity and symmetry, the necessary and sufficient condition for HSSM to be GNSD in the two-good case is that the Hicksian own-price effect (HOPE) of either of the two goods is uniformly non-positive (UNP) for all possible shares of the chosen good.[3] By focusing on the conditions for HOPE to be UNP in the two-good case, this paper offers an alternative analytical framework to the mapping of the "regular region" as done by Caves and Christensen (1980) and by Barnett and Lee (1985).

[2]The additivity and homogeneity restrictions on demand functions follow by Shephard's lemma and by Euler's theorem from the fact that the expenditure function from expenditure minimization is linearly homogeneous in prices. Also, the expenditure function is concave. Concavity implies the restriction that HSSM be GNSD because it is the Hessian matrix of the expenditure function, considering the fact that the Hessian of a concave function is negative semi–definite. That these restrictions be satisfied "globally" means satisfying them for all sets of prices and expenditures that generate non–negative quantities demanded or, equivalently, for all non–negative expenditure shares.

[3]Symmetry of the HSSM implies that all its eigenvalues are real (Sydsaeter, 1981). It is is negative semi-definite if and only if all its eigenvalues are non-positive, one of which is zero because the HSSM is singular (Chiang, 1984; Searle, 1982; Strang, 1980;). If there are only two goods, one eigenvalue is zero because of singularity and the other is non-positive if and only if the Hicksian own-price effects, which are the diagonal elements of the HSSM, are non-positive. However, because of symmetry, it is necessary and sufficient to show that the own-price effect is non-positive for only one of the two-goods.

The parameter constraints for HOPE to be UNP limit the range of possible values of the Marshallian price and income elasticities, since the Hicksian price effects are precisely related by the Slutsky equation to the Marshallian elasticities with respect to own–price (E_{ii}), cross–price (E_{ij}) and expenditure or income (E_{iI}). In some models, there are also important implications on the underlying structure of preferences. The findings in section III may be summarized as follows:

Generalized Logit: A sufficient but not necessary condition for HOPE to be UNP is that all parameters are non-negative. In this case, E_{ii} could be greater than, less than or equal to -1 (inelastic, elastic or unit elastic); E_{ij} could be greater than, less than or equal to 0 (substitutes, complements or unrelated); and E_{iI} are unrestricted. This case does not preclude corner solutions to expenditure minimization so that the set of expenditure-minimizing points in the generalized logit model coincides with the entire positive quadrant.

Translog: It is necessary and sufficient that the own-price parameters be equal and positive and that the cross-price parameters, which are symmetric, be equal to the negative of the own-price parameters. The results are that E_{ii} are strictly less than -1 (elastic); E_{ij} are strictly greater than 0 (substitutes); and E_{iI} are equal to 1 (homothetic preferences). In this case, positive quantities of both goods are bought; therefore, the set of expenditure-minimizing points in the translog is smaller than the positive quadrant.

AIDS: It is necessary and sufficient that the own-price parameter is negative (or the cross-price parameter is positive) and the income parameter is zero. Hence, E_{ii} are strictly less than -1 (elastic); E_{ij} are strictly greater than 0 (substitutes); and E_{iI} are equal to 1 (homothetic preferences). However, with or without the above conditions, the AIDS parameter constraints imply that positive quantities of both goods are bought. Therefore, the set of expenditure-minimizing points in the AIDS model is smaller than the positive quadrant. These results are qualitatively identical to those for the translog.

Generalized Leontief: HOPE is UNP or, more precisely, HOPE is uniformly zero if and only if preferences are homothetic and the cross–price parameter is zero.[4] This is the case of zero Hicksian substitutability. In this case, the E_{ii} equal the negative of the expenditure shares, i. e., strictly greater than -1 (inelastic), and E_{ij} of each good equals the negative of the share of the other good, hence strictly less than 0 (complements). Because of homotheticity, E_{iI} equal 1. Since both goods are bought, corner solutions are precluded so that the set of expenditure-minimizing solutions is smaller than the positive quadrant.

Minflex Laurent: HOPE is UNP, uniformly zero in fact, if and only if preferences are homothetic and the cross–price parameters are zero, the case of zero Hicksian substitutability.[5]. The necessary and sufficient conditions also imply that E_{ii} equal the negative of the expenditure shares or strictly greater than -1 (inelastic) and E_{ij} equal the negative of the share of the other good, hence strictly less than 0 (complements). E_{iI} are equal to 1 because of homotheticity. Moreover, the set of expenditure-minimizing points is smaller than the positive quadrant. Thus, the results are qualitatively identical to those for the generalized Leontief.

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[4]In the two–good generalized Leontief model, there is only one cross–price parameter. This need not be zero for homothetic preferences; however, given homothetic preferences, it must be zero for zero Hicksian substitutability.

[5]In the two–good minflex Laurent model, there are two cross–price parameters. The cross–price parameter associated with the negative square root of the price –income ratio need to be zero, but not the other, for homothetic preferences. However, given homothetic preferences, both cross–price parameters need to be zero for Hicksian zero substituability.

Two conclusions stand out from these results. In particular, they indicate that a globally well-behaved generalized logit model encompasses more realistic cases than the other models. In general, imposing parameter constraints to achieve globally well-behaved properties severely limits the realism of standard demand models.

II. Additivity, Homogeneity and Symmetry in a Generalized Logit Model of Expenditure Shares

Let the prices and corresponding quantities at any time period t be given by p_{it} and x_{it} , i = 1,2, . . . , n. Given that expenditures or income in the same period is I_t , then by definition of the budget constraint,

$$w_{it} = \frac{p_{it}\, x_{it}}{I_t} \quad ; \quad 1 \geq w_{it} \geq 0 \quad ; \quad \sum_{i=1}^{n} w_{it} = 1 \,. \qquad [1]$$

In [1], w_{it} is the expenditure share of each commodity. In order to satisfy [1], define a logit specification of these shares,

$$w_{it} = \frac{e^{f_{it}}}{e^{f_{1t}} + \ldots + e^{f_{nt}}} = \frac{e^{f_{it}}}{\sum_{j=1}^{n} e^{f_{jt}}} \qquad [2]$$

where f_{it} is a function of p_{it} , i = 1,2,. . . ,n and I_t . This specification guarantees non–negative predictions no matter how f_{it} is specified.[6] By defining the share w_{nt} of an arbitrarily chosen nth good in accordance with [2], it follows that

$$\ln\left(\frac{w_{it}}{w_{nt}}\right) = f_{it} - f_{nt} \quad ; \quad i = 1,2, \ldots, n-1 \,. \qquad [3]$$

Thus, the non-linear expenditure system evident from [2] can be estimated as

[6]In contrast, other models can predict non–sensical negative shares. For example, Lutton and LeBlanc (1984) showed that the translog can generate negative share predictions.

a linear system in [3] by specifying f_{it} as a linear function in [5] below.

Engel aggregation or additivity is satisfied by the logit model from the fact that [2] strictly satisfies [1] for every set of predicted shares. For Marshallian homogeneity, consider that by equating w_{it} in [1] to that in [2],

$$\ln x_{it} = -\ln\left(\frac{p_{it}}{I_t}\right) + f_{it} - \ln \sum_{j=1}^{n} e^{f_{jt}} \qquad [4]$$

The demand functions x_{it} in [4] are homogeneous of degree zero in prices and income given the following specification,

$$f_{it} = \alpha_{io} + \sum_{j=1}^{n} \alpha_{ij}\, \theta_{ij(t-1)} \ln\left(\frac{p_{jt}}{p_{it}}\right) + \beta_i \ln\left(\frac{I_t}{p_{it}}\right) \qquad [5]$$

where α_{io} and α_{ij} are parameters; and $\theta_{ij(t-1)}$ is a lagged variable weight which will be shown later to determine global Hicksian symmetry.

It follows from [2] and [4] that, in general, the own–price, cross–price and income elasticities in this logit model are

$$E_{iit} = \frac{\partial \ln x_{it}}{\partial \ln p_{it}} = -1 + \frac{\partial f_{it}}{\partial \ln p_{it}} - \sum_{j=1}^{n} w_{jt} \frac{\partial f_{jt}}{\partial \ln p_{it}} \qquad [6]$$

$$E_{ikt} = \frac{\partial \ln x_{it}}{\partial \ln p_{kt}} = \frac{\partial f_{it}}{\partial \ln p_{kt}} - \sum_{j=1}^{n} w_{jt} \frac{\partial f_{jt}}{\partial \ln p_{kt}} \quad ; \quad k \neq i \qquad [7]$$

$$E_{iIt} = \frac{\partial \ln x_{it}}{\partial \ln I_t} = 1 + \frac{\partial f_{it}}{\partial \ln I_t} - \sum_{j=1}^{n} w_{jt} \frac{\partial f_{jt}}{\partial \ln I_t} \; . \qquad [8]$$

Substituting the appropriate derivatives of f_{it} in [5] into the elasticity formulas in [6], [7] and [8], it can be verified that

$$E_{iit} = -1 - (1 - w_{it})\left(\sum_{k=1}^{n} \alpha_{ik}\, \theta_{ik(t-1)} + \beta_i\right) - \sum_{k=1}^{n} w_{kt}\, \alpha_{ki}\, \theta_{ki(t-1)} \; ; \qquad [9]$$

$$E_{ikt} = \alpha_{ik}\,\theta_{ik(t-1)} - \sum_{i=1}^{n} w_{it}\,\alpha_{ik}\,\theta_{ik(t-1)} + w_{kt}\left(\sum_{i=1}^{n} \alpha_{ki}\,\theta_{ki(t-1)} + \beta_k\right) ; \qquad [10]$$

$$E_{iIt} = 1 + (1 - w_i)\,\beta_i - \sum_{k=1}^{n} w_{kt}\,\beta_k \quad ; \quad i \neq k \ . \qquad [11]$$

It can be verified from these elasticities that for any good,

$$\sum_{j=1}^{n} E_{ijt} + E_{iIt} = 0 \ . \qquad [12]$$

The result in [12] means that the logit specification satisfies Marshallian zero–degree homogeneity in prices and income.[7]

One disadvantage of direct specification of a demand system, as opposed to deriving it from an expenditure function or indirect utility function, is the need to check for symmetry (Lau, 1976).[8] However, Hicksian symmetry of the cross–price effects can be obtained in the logit model above by defining θ as the following function of lagged shares and by imposing symmetry on the α coefficients,

$$\theta_{ik(t-1)} = \frac{w^{\gamma}_{k(t-1)}}{w^{1-\gamma}_{i(t-1)}} \quad ; \quad w_{i(t-1)} = \frac{p_{i(t-1)}\ x_{i(t-1)}}{I_{(t-1)}} \quad ; \quad \alpha_{ik} = \alpha_{ki} \qquad [13]$$

[7]For this result, it is sufficient to define Marshallian demand for any good as a function of the ratios of its own price to income and to the prices of the other goods. For example, this condition is satisfied by the model specification in [4] and [5]. This insures that proportional changes in all prices and income will leave demand unchanged, which is required by the above homogeneity property. This property implies the result in [12].

[8]If the hypothesized expenditure function is twice continuously diffferentiable with respect to prices, symmetry of the Hicksian cross–price effecsts follows by Shephard's lemma and Young's theorem. If the expenditure function is invertible, i. e., the indirect utility function is obtainable, Roy's identity yields the Marshallian demand system. From this, Hicksian symmetry can be verified by means of the Slutsky equation. Thus, symmetry is automatic for demand systems derived from expenditure functions with the above properties but not for demand systems directly specified. Hence, the need to check for symmetry in the latter case.

where γ is a parameter[9]. To show that the definitions and restrictions in [13] are sufficient for global Hicksian symmetry, consider the Slutsky equation

$$\frac{\partial x_{it}}{\partial p_{kt}} = \frac{\partial x^h_{it}}{\partial p_{kt}} - x_{kt}\frac{\partial x_{it}}{\partial I_t} \qquad [14]$$

where x^h_{it} and x^h_{kt} are Hicksian demand functions whereas x_{it} and $x_{kt}$are Marshallian. Now, the Hicksian cross-price effect in [14] can be expressed in terms of Marshallian price and income elasticities and budget shares as

$$\frac{\partial x^h_{it}}{\partial p_{kt}} = \frac{I_t}{p_{it}\ p_{kt}}\left(E_{ikt}\ w_{it} + w_{it}\ w_{kt}\ E_{iIt}\right) \qquad \text{or} \qquad [15]$$

$$\frac{\partial x^h_{kt}}{\partial p_{it}} = \frac{I_t}{p_{kt}\ p_{it}}\left(E_{kit}\ w_{kt} + w_{kt}\ w_{it}\ E_{kIt}\right)\ . \qquad [16]$$

Symmetry of the Hicksian cross–price effects holds in the generalized logit model for any set of predicted budget shares. For infinitesimal changes of shares, the time lag defined by the original data, t–1 , may be replaced by an infinitesimal lag, $t-\delta$ where $\delta \to 0$. This means that the elasticities may be computed conditionally, using on the right-hand side the shares evaluated at time t , i.e., using the current value in place of the lagged value of θ in [13]. In this case, omitting the time subscript t for simplicity, it can be verified that

$$w_i\ \theta_{ik} = w_k\ \theta_{ki}\ . \qquad [17]$$

In view of [17], the price elasticities in [9] and [10] simplify to

$$E_{ii} = -1 - \sum_{k=1}^{n} \alpha_{ik}\ \theta_{ik} - (1 - w_i)\ \beta_i \quad ; \qquad [18]$$

$$E_{ik} = \alpha_{ik}\ \theta_{ik} + w_k\ \beta_k\ . \qquad [19]$$

The income elasticities remain the same as in [11], which are,

[9]Considine's (1990) model is a special case of the generalized logit model in the sense that his global symmetry restriction may be obtained from [13] by setting $\gamma = 1$, thus eliminating the denominator of θ, and then replacing the actual lagged value of the share w_k by its predicted value. His symmetry restrictions on the price parameters are equivalent to the above restrictions on the α coeficients.

$$E_{iI} = 1 + (1 - w_i)\,\beta_i - \sum_{k=1}^{n} w_k\,\beta_k \,. \qquad [20]$$

Substituting these elasticities into [15] and [16] gives the Hicksian own–price and cross–price effects,

$$\frac{\partial x_i^h}{\partial p_i} = -\frac{I}{p_i^2}\left[w_i - w_i^2 + \sum_{k=1}^{n} \alpha_{ik}\,(w_i\,w_k)^{\gamma} + (w_i - 2\,w_i^2)\,\beta_i + w_i^2 \sum_{j=1}^{n} w_j\,\beta_j\right]; \quad [21]$$

$$\frac{\partial x_i^h}{\partial p_k} = \frac{I}{p_i\,p_k}\left[w_i\,w_k + \alpha_{ik}\,(w_i\,w_k)^{\gamma} + w_i\,w_k\,(\beta_i + \beta_k) - w_i\,w_k \sum_{j=1}^{n} w_j\,\beta_j\right]; \quad [22]$$

$$\frac{\partial x_k^h}{\partial p_i} = \frac{I}{p_k\,p_i}\left[w_k\,w_i + \alpha_{ki}\,(w_k\,w_i)^{\gamma} + w_k\,w_i\,(\beta_k + \beta_i) - w_k\,w_i \sum_{j=1}^{n} w_j\,\beta_j\right]. \quad [23]$$

The Hicksian cross–price effects in [22] and [23] are symmetric given $\alpha_{ik} = \alpha_{ki}$ in [13]. Global symmetry holds for every set of shares and for any value of γ.

The global symmetry result above is conditional on the equality between the "approximate" short–run elasticities in [18] to [20] and the "true" short–run elasticities in [9] to [11]. These elasticities are equal given the assumption of fixed expenditure shares in [13], which is reasonable in the short–run. Granted this assumption, the global symmetry between [22] and [23] implies that there exists in principle an underlying expenditure function or a dual indirect utility function that could generate the logit model (Samuelson, 1950; Katzner, 1970; Hurwicz and Uzawa, 1971; Johnson, Hassan and Green, 1984; Varian, 1984; LaFrance and Haneman, 1989).[10]

[10] By Shephard's lemma, the Hicksian demand functions are the first partial derivatives of the expenditure function with respect to prices. Thus, these demand functions form a system of partial differential equations. This system is integrable or has a solution, i. e., the expenditure function exists, if and only if the first order cross–partial derivatives of the system are symmetric. This integrability condition is equivalent to the symmetry of the second–order partial derivatives with respect to prices of the expenditure function, which is true by Young's theorem. Thus, the absence of symmetry not only violates Young's theorem but also implies that the Hicksian demand system is not integrable or that the expenditure function and its dual indirect utility function do not exist.

It may be noted that the generalized logit model satisfies the definition of a flexible functional form model in that, given an appropriate set of the parameters α_{ij} and β_i in [5], it is capable of producing the full range of price and income elasticities at each data point in a sample of observations (Caves and Christensen, 1980). It also satisfies the requirement on the minimum number of free parameters in order to be a flexible functional form (Diewert and Wales, 1987).[11]

Finally, there is a theoretical attribute of the generalized logit elasticities that has important implications for empirical work. It is in the nature of econometric estimates using aggregate data that the estimated parameters apply not only to the aggregate economic unit as a whole but also to the groups or subgroups subsumed under the aggregate. However, it is not simply interesting but quite important in applied demand analysis to be able to distinguish between group differences in demand for the same commodity.

An interesting example is the difference in demand elasticities for electricity between households that use either electricity, natural gas or fuel oil for space heating. This can easily be handled by the logit elasticities in [18] to [20]. For instance, since by definition an all–electric household does not buy natural gas or fuel oil, then the own–price, cross–price and income elasticities for electricity of this household can be computed simply by setting to zero the expenditure share of natural gas, fuel oil and the share of any good included in the demand system which the household does not buy.

In general, differences in demand elasticities for the same good between

[11]The generalized logit has to satisfy the following equations, namely, 1 budget constraint, N demand functions, N^2 partial derivatives with respect to prices and N partial derivatives with respect to income. The number of restrictions are 1 for additivity, N for homogeneity, $(N^2 - N)/2$ symmetry restrictions on parameters ($\alpha_{ij} = \alpha_{ji}$) and N zero restrictions ($\alpha_{ii} = 0$). Thus, the minimum number of free parameters equals the difference between the total number of equations and the total number of restrictions and this difference equals $(N^2 + N)/2$. This is exactly equal to the number of parameters estimated in the generalized logit model since there are $(N^2 - N)/2$ unique α_{ij} coefficients and N different β_i coefficients.

diverse groups can be determined from the fact that in the generalized logit model, $\theta_{ik} = 0$ whenever $w_k = 0$. One of the interesting results is that whenever this is true, $E_{ik} = 0$, which means that the elasticity of demand by an individual (or by any grouping of individuals) for any good i that it buys is unaffected by the change in the price of any other good k that it does not buy. This result comes out naturally from the generalized logit but is not exhibited by the other demand models.

III. Negative Semi–Definiteness of The HSSM With Two Goods

Hicksian price effects are not directly measurable because Hicksian demand functions are not observable. However, these Hicksian price effects can be measured exactly from observable Marshallian demand functions by means of the Slutsky equation. Thus, every element of the HSSM can be computed without knowing the Hicksian demand functions.

The Slutsky equation satisfies Euler's theorem that Hicksian demand functions are homogeneous of degree zero in prices. This follows from [15] since for all i, j = 1, 2, . . ., n,

$$\sum_{j=1}^{n} \frac{\partial x_i^h}{\partial p_j} p_j = \frac{I}{p_i} w_i \left(\sum_{j=1}^{n} E_{ij} + E_{iI} \sum_{j=1}^{n} w_j \right) = 0 \qquad [24]$$

That [24] equals zero as required by Euler's theorem is true since

$$\sum_{j=1}^{n} w_j = 1 \quad \text{and} \quad \sum_{j=1}^{n} E_{ij} + E_{iI} = 0 \ . \qquad [25]$$

Thus, additivity and Marshallian zero–degree homogeneity in prices and income imply Hicksian zero–degree homogeneity in prices.[12]

Let s_{ij} be the elements of HSSM, i. e.,

$$HSSM = \{s_{ij}\} \quad ; \quad s_{ij} = \frac{\partial x_i^h}{\partial p_j} \quad ; \quad i,j = 1,2,\ldots,n \,. \qquad [26]$$

For n = 2 , [24] and [26] yield

$$s_{11}\, p_1 + s_{12}\, p_2 = 0 \quad ; \quad \text{and} \qquad [27]$$

$$s_{21}\, p_1 + s_{22}\, p_2 = 0 \;. \qquad [28]$$

Symmetry implies that $s_{12} = s_{21}$. In this case, it follows from [27] and [28] that

$$s_{11} \leq 0 \quad \Leftrightarrow \quad s_{12} = s_{21} \geq 0 \quad \Leftrightarrow \quad s_{22} \leq 0 \;. \qquad [29]$$

The result in [29] is the standard conclusion in a two–good case that a non–positive Hicksian own–price effect (HOPE) for one good is necessary and sufficient for the own–price effect to be non–positive for the other good as well. Moreover, the two goods can only be Hicksian (net) substitutes since the symmetric cross–price effect must be non–negative.[13] Finally, since prices are positive,i.e., $p_1 > 0$ and $p_2 > 0$, [27] and [28] imply that HSSM is singular or that its determinant (det) equals zero, i.e.,

$$\det(HSSM) = s_{11}\, s_{22} - s_{12}^2 = 0 \;. \qquad [30]$$

To show that HSSM is negative semi–definite, let λ be a scalar representing its eigenvalue and I be an identity matrix of the same order as HSSM. It follows from [30] that the characteristic equation is given by

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[12]All of the restrictions on demand functions can be derived without reference to a utility function and simply follow from the linear homogeneity and concavity properties of the expenditure function (McKenzie, 1957; Takayama, 1985). Negative semi–definiteness of the HSSM is a weaker and more general condition than quasi–concavity of the utility function because the expenditure function can be concave even if the utility function is not quasi–concave, i. e., negative semi–definiteness does not imply quasi–concavity (Deaton and Muellbauer, 1980b). However, negative semi–definiteness is implied by quasi–concavity (Phlips, 1987).

[13]The results in [29] include the possibility that the equality to zero holds. This is true for example in interior solutions to expenditure minimization given a Leontief type utility function. This could also be true at corner solutions in the quantity axis when the utility function is not quasi–concave.

$$\det(\text{HSSM} - \lambda I) = -(s_{11} + s_{22})\,\lambda + \lambda^2 = 0 \ . \qquad [31]$$

From [29] and [31],

$$-(s_{11} + s_{22})\,\lambda + \lambda^2 = 0 \ \Rightarrow \ (\lambda_1 = 0 \ \text{ and } \ \lambda_2 \le 0) \ \text{ if } \ (s_{11} \le 0 \ \text{ and } \ s_{22} \le 0) \quad [32]$$

but, on the contrary,

$$-(s_{11} + s_{22})\,\lambda + \lambda^2 = 0 \ \Rightarrow \ (\lambda_1 = 0 \ \text{ and } \ \lambda_2 > 0) \ \text{ if } \ (s_{11} > 0 \ \text{ and } \ s_{22} > 0). \quad [33]$$

But [33] contradicts the definition of a negative semi–definite matrix that it has non–positive eigenvalues (Strang, 1980). It follows that a non–positive HOPE, i. e., $s_{11} \le 0$, is necessary and sufficient for one eigenvalue to be zero and the other to be non–positive, which is necessary and sufficient for HSSM to be negative semi–definite in the two–good case (Chiang, 1984).

It follows from [12] and [15] that, in general, the HOPE equation for good 1 in the two–good case can be written as

$$s_{11} = \frac{\partial x_1^h}{\partial p_1} = \frac{I}{p_1^2}\, w_1\,[(1 - w_1)\,E_{11} - w_1\,E_{12}] \ . \qquad [34]$$

Hence, global negative semi–definiteness can be imposed on the HSSM by imposing parameter restrictions on the demand model such that HOPE is uniformly non–positive (UNP) in the above case for all non–negative expenditure shares. However, it is clear from [34] that such restrictions must limit the range of values not only of the price elasticities but also of the income elasticities, since the sum of all elasticities equals zero for each good. These issues are explored in the following analysis of the generalized logit, translog, AIDS, generalized Leontief and minflex Laurent demand systems.

III – (A) The Generalized Logit Model

Suppose that there are only two goods. In this case, it follows from [18] to [20] that the price and income elasticities for good 1 are

$$E_{11} = -1 - \alpha_{12}\,\theta_{12} - (1 - w_1)\,\beta_1 \quad ; \qquad [35]$$

$$E_{12} = \alpha_{12}\,\theta_{12} + (1 - w_1)\,\beta_2 \quad ; \quad \text{and} \qquad [36]$$

$$E_{1I} = 1 + (1 - w_1)\,(\beta_1 - \beta_2)\ . \qquad [37]$$

These elasticities sum to zero because of zero–degree homogeneity in prices and income. Combining [35], [36] and [34] yields the HOPE equation

$$s_{11} = -\frac{I}{p_1^2}\,[(w_1 - w_1^2) + (w_1 - w_1^2)^{\gamma}\,\alpha_{12} + w_1\,(1 - w_1)^2\,\beta_1 + w_1^2\,(1 - w_1)\,\beta_2]\ . \quad [38]$$

It follows that $s_{11} \leq 0$ if and only if

$$[(w_1 - w_1^2) + (w_1 - w_1^2)^{\gamma}\,\alpha_{12} + w_1\,(1 - w_1)^2\,\beta_1 + w_1^2\,(1 - w_1)\,\beta_2] \geq 0\ . \qquad [39]$$

Since $1 \geq w_1 \geq 0$, then all the terms involving w_1 are non–negative. Therefore, a sufficient condition for [39] is that all parameters are non–negative, namely, $\alpha_{12} \geq 0$, $\beta_1 \geq 0$ and $\beta_2 \geq 0$. However, these parameter restrictions are not necessary in order for [39] to be true globally. To show this, suppose that $\gamma = 1$. In this case, $\alpha_{12} \geq -1$, $\beta_1 \geq 0$ and $\beta_2 \geq 0$ are sufficient.

Therefore, in order for $s_{11} \leq 0$, it is possible from [35] to [37] that E_{11} could be greater than, less than or equal to -1; E_{12} could be greater than, less than or equal to 0; and E_{1I} could be greater than, less than or equal to 1. These apply as well to the elasticities E_{22}, E_{21} and E_{2I} of good 2.

III – (B) The Translog Model

The budget shares of the Translog model (Christensen, Jorgenson and Lau, 1975) can be written as

$$w_i = \frac{\beta_{io} + \sum_{j=1}^{n} \beta_{ij} \ln\left(\frac{p_j}{I}\right)}{\sum_{i=1}^{n} \beta_{io} + \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_{ij} \ln\left(\frac{p_j}{I}\right)} \quad ; \quad \beta_{ij} = \beta_{ji} \quad ; \quad \sum_{i=1}^{n} \beta_{io} = -1\ . \qquad [40]$$

In the two–good case of [40], the price and income elasticities of good 1 are

$$E_{11} = -1 + \frac{1}{D}\left[\left(\frac{1-w_1}{w_1}\right)\beta_{11} - \beta_{12}\right] \quad ; \qquad [41]$$

$$E_{12} = \frac{1}{D}\left[\left(\frac{1-w_1}{w_1}\right)\beta_{12} - \beta_{22}\right] \quad ; \quad \text{and} \qquad [42]$$

$$E_{1I} = 1 - \frac{1}{D}\left[\left(\frac{1-w_1}{w_1}\right)(\beta_{11} + \beta_{12}) - \beta_{12} - \beta_{22}\right] \qquad [43]$$

where

$$D = -1 + (\beta_{11} + \beta_{12}) \ln\left(\frac{p_1}{I}\right) + (\beta_{12} + \beta_{22}) \ln\left(\frac{p_2}{I}\right) . \qquad [44]$$

The price and income elasticities in [41] to [43] sum to zero because of homogeneity. These can be combined with [34] to obtain the HOPE equation

$$s_{11} = -\frac{I}{p_1^2}\left\{(1-w_1)\, w_1 - \frac{1}{D}\left[\beta_{11}(1-w_1)^2 - 2\,\beta_{12}(1-w_1)\, w_1 + \beta_{22}\, w_1^2\right]\right\} . \quad [45]$$

In evaluating [45] when both goods are bought, note that s_{11} assumes that utility is held constant, by definition of a Hicksian demand function. This means that I must be adjusted in the opposite direction to the change in p_1 in order to return to the original indifference curve. Given the price of the other good, p_2, it follows that D in [44] can switch signs as p_1 and I change from the fact that D is a function of the logarithmn of the ratio of prices to income. For strictly positive prices and income, this ratio varies from less than to greater than one so that the logarithmn of this ratio changes from negative to positive, thus changing the sign of D and of s_{11} as well. To prevent this change in sign from happening, it is necessary that

$$\beta_{11} + \beta_{12} = 0 \quad \text{and} \quad \beta_{12} + \beta_{22} = 0 . \qquad [46]$$

Hence, the Translog HOPE equation in [45] becomes

$$s_{11} = - \frac{I}{p_1^2} \left[(1 - w_1)\, w_1 + \beta_{11} \right] \qquad [47]$$

where β_{11} cannot be zero when w_1 is positive. This shows that [46], while necessary, is not sufficient for [47] to be non–positive. Notice that [46] means that β_{22} is not equal to zero since β_{11} cannot be zero. This presumes an interior solution in which $1 > w_1 > 0$. In this case, the necessary and sufficient condition is

$$\beta_{11} = \beta_{22} = - \beta_{12} > 0 \ . \qquad [48]$$

The sufficiency of [48] is obvious from [47]. Necessity follows from the fact that if $\beta_{11} < 0$ then w_1 can always be made sufficiently close to zero such that [47] yields $s_{11} > 0$, which is a contradiction.

However, it follows from [48] that [41] to [43] yield

$$E_{11} = -1 - \frac{\beta_{11}}{w_1} < -1 \quad ; \quad E_{12} = \frac{-\beta_{12}}{w_1} > 0 \quad ; \quad \text{and} \quad E_{1I} = 1 \ . \qquad [49]$$

These results mean that good 1 cannot be price–inelastic, cannot be a gross complement and must have a unitary income elasticity. This applies to good 2 as well, i. e., $E_{22} < -1$, $E_{21} > 0$ and $E_{2I} = 1$. Thus, the implied underlying utility function must be homothetic. These results indicate that a globally well–behaved translog rules out a lot of realistic cases.

III – (C) The AIDS Model

The budget shares of the "almost ideal demand system" (AIDS) (Deaton and Muellbauer, 1980a, 1980b) can be written as

$$w_i = \delta_i + \sum_{j=1}^{n} \gamma_{ij} \ln p_j + \tau_i \ln \left(\frac{I}{P} \right) \quad ; \qquad [50]$$

where P is a price index defined by

$$\ln P = \delta_0 + \sum_{i=1}^{n} \delta_i \ln p_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \ln p_i \ln p_j \ . \qquad [51]$$

The parameter restrictions in [50] and [51] are that for all j,

$$\sum_{i=1}^{n} \delta_i = 1 \quad ; \quad \sum_{i=1}^{n} \tau_i = 0 \quad ; \quad \sum_{i=1}^{n} \gamma_{ij} = 0 \quad ; \quad \sum_{j=1}^{n} \gamma_{ij} = 0 \quad ; \quad \gamma_{ij} = \gamma_{ji} \ . \qquad [52]$$

The first three restrictions in [52] are for additivity; the fourth is for homogeneity; and the fifth is for symmetry.

For two goods, the AIDS model yields for good 1 the following price and income elasticities,[14]

$$E_{11} = -1 + \frac{1}{w_1} \left\{ \gamma_{11} - \tau_1 \left[w_1 - \tau_1 \ln \left(\frac{I}{P} \right) \right] \right\} \quad ; \qquad [53]$$

$$E_{12} = \frac{1}{w_1} \left\{ \gamma_{12} - \tau_1 \left[w_2 - \tau_2 \ln \left(\frac{I}{P} \right) \right] \right\} \quad ; \quad \text{and} \qquad [54]$$

$$E_{1I} = 1 + \frac{\tau_1}{w_1} \ . \qquad [55]$$

Given the parameter restrictions in [52], the sum of these elasticities equals zero as required by homogeneity. Substituting [53] and [54] into the expression for HOPE in [34] gives

$$s_{11} = -\frac{I}{p_1^2} \left[(1 - w_1)\, w_1 - \gamma_{11} - \tau_1^2 \ln \left(\frac{I}{P} \right) \right] . \qquad [56]$$

The above results are defined if good 1 is bought, so that γ_{11}, γ_{12} and τ_1 are not equal to zero at the same time. With only two goods, the parameter restrictions imply that $\gamma_{11} = -\gamma_{12}$, $\gamma_{21} = -\gamma_{22}$, $\gamma_{12} = \gamma_{21}$, and $\tau_1 = -\tau_2$. By implication, both goods must be bought in which case $1 > w_i > 0$. Thus, γ_{ij} and τ_i cannot both be zero, although one of these parameters could be zero.

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[14]For an analysis of the relationships between the above AIDS elasticities and the AIDS elasticities obtained if [51] is replaced by a Stone price index refer to Green and Alston, 1990.

From [56], the necessary and sufficient condition for $s_{11} \leq 0$ is

$$\left[(1 - w_1)\, w_1 - \gamma_{11} - \tau_1^2 \ln\left(\frac{I}{P}\right)\right] \geq 0 \qquad [57]$$

for all $1 > w_1 > 0$. However, it is clear that this condition cannot be satisfied unless $\tau_1 = 0$ since the sign of the logarithmn of the ratio of I to P will switch from positive, negative or zero for all values of I and P. Therefore, although not sufficient, it is necessary that $\tau_1 = \tau_2 = 0$. In this case, $\gamma_{11} = -\gamma_{12}$ cannot be zero since, as previously observed, both goods must be bought. It follows that for a globally well–behaved AIDS model with two goods, the necessary and sufficient conditions are

$$\tau_1 = \tau_2 = 0 \quad \text{and} \quad \gamma_{11} = -\gamma_{12} < 0 \ . \qquad [58]$$

Given $\tau_1 = 0$, the sufficiency of $\gamma_{11} < 0$ is obvious from [57]. Since γ_{11} cannot be zero, the contrary case is that $\gamma_{11} > 0$. In this case, w_1 can be made sufficiently close to 1 such that [57] is contradicted. This establishes necessity.

It follows from [58] that the AIDS price and income elasticities for good 1 in [53] to [55] become

$$E_{11} = -1 + \frac{\gamma_{11}}{w_1} < -1 \quad ; \quad E_{12} = \frac{\gamma_{12}}{w_1} > 0 \quad ; \quad \text{and} \quad E_{1I} = 1 \ . \qquad [59]$$

These restrictions on the AIDS elasticities are similar to the restrictions on the elasticities of a globally well–behaved Translog shown in [49]. However, since both goods must be bought, [59] applies to good 2 as well, i. e., $E_{22} < -1$, $E_{21} > 0$ and $E_{2I} = 1$. These results rule out a lot of realistic cases because the goods cannot be price–inelastic, cannot be gross complements and must have unitary income elasticities. The latter implies that the underlying preferences must be homothetic in a globally well–behaved AIDS model.

III – (D) The Generalized Leontief Consumer Demand System

It is shown in Christensen and Caves (1980) and in Barnett and Lee (1985) that the budget share in a generalized Leontief model is

$$w_i = \frac{\delta_i \left(\frac{p_i}{I}\right)^{\frac{1}{2}} + \sum_{j=1}^{n} \gamma_{ij} \left(\frac{p_i}{I}\right)^{\frac{1}{2}} \left(\frac{p_j}{I}\right)^{\frac{1}{2}}}{\sum_{i=1}^{n} \delta_i \left(\frac{p_i}{I}\right)^{\frac{1}{2}} + \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij} \left(\frac{p_i}{I}\right)^{\frac{1}{2}} \left(\frac{p_j}{I}\right)^{\frac{1}{2}}} \qquad [60]$$

where I is income; p_i and p_j are prices; δ_i and γ_{ij} are parameters. It is assumed that $\gamma_{ij} = \gamma_{ji}$. For the case of two goods, let

$$R_1 = \frac{p_1}{I} \quad ; \quad R_2 = \frac{p_2}{I} \quad ; \quad S_1 = R_1^{\frac{1}{2}} \quad ; \quad S_2 = R_2^{\frac{1}{2}} \quad ; \qquad [61]$$

$$A_1 = \delta_1 S_1 + \gamma_{11} R_1 + \gamma_{12} S_1 S_2 \quad ; \quad \text{and} \qquad [62]$$

$$A_2 = \delta_2 S_2 + \gamma_{22} R_2 + \gamma_{12} S_1 S_2 \ . \qquad [63]$$

It follows from the above that the share of good 1 can be expressed as

$$w_1 = \frac{A_1}{B} \quad ; \quad B = A_1 + A_2 \qquad [64]$$

so that the price and income elasticities are

$$E_{11} = -1 + (\delta_1 S_1 + 2\gamma_{11} R_1)\left(\frac{1 - w_1}{2 w_1 B}\right) + \gamma_{12} S_1 S_2 \left(\frac{1 - 2 w_1}{2 w_1 B}\right) \quad ; \qquad [65]$$

$$E_{12} = \gamma_{12} S_1 S_2 \left(\frac{1 - 2 w_1}{2 w_1 B}\right) - (\delta_2 S_2 + 2\gamma_{22} R_2) \frac{1}{2 B} \quad ; \quad \text{and} \qquad [66]$$

$$E_{1I} = 1 - (\delta_1 S_1 + 2\gamma_{11} R_1)\left(\frac{1 - w_1}{2 w_1 B}\right) - \gamma_{12} S_1 S_2 \left(\frac{1 - 2 w_1}{w_1 B}\right)$$

$$+ (\delta_2 S_2 + 2\gamma_{22} R_2) \frac{1}{2 B} \ . \qquad [67]$$

The sum of the above elasticities equals zero because of homogeneity.

By substituting [65] and [66] into [34] the HOPE equation in the generalized Leontief model can be written as

$$s_{11} = -\frac{I}{p_1^2}\left\{(1-w_1)\,w_1 - (\delta_1\,S_1 + 2\,\gamma_{11}\,R_1)\,\frac{(1-w_1)^2}{2\,B}\right.$$

$$\left. - \gamma_{12}\,S_1\,S_2\,\frac{(1-2\,w_1)^2}{2\,B} - (\delta_2\,S_2 + 2\,\gamma_{22}\,R_2)\,\frac{w_1^2}{2\,B}\right\}. \qquad [68]$$

In evaluating [68], it is enlightening to note from [61] to [64] that the requirement that w_1 is non–negative does not impose specific signs on the parameters. Consider the case where $p_1 = p_2 = I$. In this case, it is only necessary that the sign of $(\delta_1 + \gamma_{11} + \gamma_{12})$ be the same as tha sign of $(\delta_2 + \gamma_{22} + \gamma_{12})$. Given a set of parameter values that satisfy this condition, it is possible in [68] for s_{11} to switch signs as w_1 varies in the interval $1 \geq w_1 \geq 0$ when p_1, p_2 and I change.[15] This implies that without additional parameter restrictions HOPE is not UNP in a two–good generalized Leontief model.

Homothetic preferences and zero Hicksian substitutability is a clear cut case when the generalized Leontief is globally well–behaved. To show this, let $\delta_1 = \delta_2 = 0$. Thus, income can be factored out of the share equation, making shares invariant with respect to income as implied by homotheticity. It can be verified that in the homothetic generalized Leontief, the own–price and cross–price elasticities are

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[15]For example, let $\delta_1 = \delta_2 = 0.025$, $\gamma_{11} = \gamma_{22} = 0.50$ and $\gamma_{12} = -\,0.01$. Also, let p_1 increase from 1 to 25 as p_2 decreases from 25 to 1. As this happens, p_1 and p_2 will equal each other at 13. Since the parameters are equal for the two goods, then when $p_1 = p_2 = 13$, the share $w_1 = 0.50$ when $I = 13$. Keeping $I = 13$ as p_1 and p_2 change above, it can be verified that w_1 increases from 0.021883 to 0.997506. Correspondingly, s_{11} will increase from $-$ 0.01297 (when $p_1 = 1$, $p_2 = 25$ and $I = 13$) to 0.0 and then to 0.000007 before it falls to 0.000005 (when $p_1 = 25$, $p_2 = 1$ and $I = 13$). Qualitatively, these results can be obtained if the level of I is also variable. In this example, s_{11} in [68] clearly switches signs with $1 > w_1 > 0$, thus showing that the generalized Leontief is not globally well–behaved.

$$E_{11} = -1 + \frac{1}{G}\left[\gamma_{11}\left(\frac{p_1}{p_2}\right)\left(\frac{1-w_1}{w_1}\right) + \gamma_{12}\left(\frac{p_1}{p_2}\right)^{\frac{1}{2}}\left(\frac{1-2\,w_1}{2\,w_1}\right)\right] \quad ; \quad \text{and} \qquad [69]$$

$$E_{12} = -\frac{1}{G}\left[\gamma_{11}\left(\frac{p_1}{p_2}\right)\left(\frac{1-w_1}{w_1}\right) + \gamma_{12}\left(\frac{p_1}{p_2}\right)^{\frac{1}{2}}\left(\frac{1-2\,w_1}{2\,w_1}\right)\right] \qquad [70]$$

where

$$w_1 = \frac{F_1}{G} \;\; ; \;\; F_1 = \gamma_{11}\left(\frac{p_1}{p_2}\right) + \gamma_{12}\left(\frac{p_1}{p_2}\right)^{\frac{1}{2}} \;\; ; \;\; G = \gamma_{11}\left(\frac{p_1}{p_2}\right) + 2\,\gamma_{12}\left(\frac{p_1}{p_2}\right)^{\frac{1}{2}} + \gamma_{22}\,. \qquad [71]$$

Combining [69], [70] and [34] yields the HOPE equation in the homothetic generalized Leontief,

$$s_{11} = -\frac{I}{p_1^2}\left\{(1-w_1)\,w_1 - \frac{1}{G}\left[\gamma_{11}\,(1-w_1)\left(\frac{p_1}{p_2}\right) + \gamma_{12}\left(\frac{1-2\,w_1}{2}\right)\left(\frac{p_1}{p_2}\right)^{\frac{1}{2}}\right]\right\}. \qquad [72]$$

In evaluating [72], consider the case where $p_1 = p_2$ in which non–negativity of w_1 implies from [71] that the sign of $(\gamma_{11} + \gamma_{12})$ be the same as the sign of $(\gamma_{22} + \gamma_{12})$. Thus, suppose γ_{11} is positive. Then, no matter what the sign of γ_{12}, it is possible that s_{11} in [72] will switch signs as w_1 changes between 1 and 0. This shows that homotheticity is not sufficient for HOPE to be UNP in a generalized Leontief model.

However, homothetic preferences ($\delta_1 = \delta_2 = 0$) and a zero cross–price parameter ($\gamma_{12} = 0$) are sufficient since it follows then from [71] and [72] that

$$s_{11} = -\frac{I}{p_1^2}\,\{(1-w_1)\,w_1 - (1-w_1)\,w_1\} = 0 \quad \text{for all } w_1. \qquad [73]$$

The uniformly zero value of s_{11} in [73] implies from [27] that $s_{12} = 0$, i. e., zero Hicksian substitutability.[16] For this result, homotheticity and a zero cross–price parameter are also necessary since, otherwise, [73] cannot be uniformly zero.

Finally, from [69] to [71], it follows that since $\gamma_{12} = 0$,

[16] In this special case, the HSSM is a null matrix, which is trivially negative semi–definite because it has zero eigenvalues.

$$E_{11} = -w_1 > -1 \quad ; \quad E_{12} = -(1-w_1) < 0 \quad ; \quad E_{1I} = 1 \ . \qquad [74]$$

Since these hold for good 2 as well, then in a globally well–behaved generalized Leontief characterized by homothetic preferences and zero Hicksian substitutability, the demands for goods are inelastic and the goods are (gross) complements with unitary income elasticities.[17]

III – (E) The Minflex Laurent Consumer Demand System

From Barnett and Lee (1985), the budget share in a minflex Laurent demand system is

$$w_i = \frac{a_i z_i + a_{ii} v_i + \sum_{j:j\neq i} a_{ij}^2 z_i z_j + \sum_{j:j\neq i} b_{ij}^2 \bar{z}_i \bar{z}_j}{a' z + \sum_k a_{kk} v_k + \sum\sum_{(j,k)\in S} a_{jk}^2 z_j z_k + \sum\sum_{(j,k)\in S} b_{jk}^2 \bar{z}_j \bar{z}_k} \qquad [75]$$

where, by definition,[18]

$$a' = (a_1, \ldots, a_n) \quad ; \quad z = (z_1, \ldots, z_n) \quad ; \quad z_i = v_i^{\frac{1}{2}} \quad ; \quad \bar{z}_i = z_i^{-1} \quad ;$$

$$p' = (p_1, \ldots, p_n) \quad ; \quad x = (x_1, \ldots, x_n)' \quad ; \quad I = p' x \quad ; \quad \text{and} \quad v_i = \frac{p_i}{I} \ .$$

The coefficients denoted by a and b define the demand parameters, where a_{ij} and b_{ij} are symmetric, i. e., $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$. Given only two goods, let

$$M_1 = a_1 z_1 + a_{11} v_1 + a_{12}^2 z_1 z_2 + b_{12}^2 \bar{z}_1 \bar{z}_2 \quad ; \quad \text{and} \qquad [76]$$

$$M_2 = a_2 z_2 + a_{22} v_2 + a_{21}^2 z_2 z_1 + b_{21}^2 \bar{z}_2 \bar{z}_1 \ . \qquad [77]$$

Therefore, the budget share of good 1 is

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[17]Gross substitutability or complementarity is a Marshallian concept allowing utility to change as price changes, as opposed to net substitutability or complementarity whch is Hicksian, thus requiring utility to be fixed.

[18]Barnett and Lee denoted the budget share by s_i, which is replaced by w_i in [71] for consistency with earlier notation for the budget share used throughout this paper. For this reason, the terms in Barnett and Lee's paper denoted originally by w_i were replaced above by z_i. Moreover, m and q were replaced by I and x to denote income and quantity, respectively. The rest of the notation are the same as in Barnett and Lee's paper.

$$w_1 = \frac{M_1}{N} \quad ; \quad N = M_1 + M_2 \tag{78}$$

and the elasticities with respect to prices and income are

$$E_{11} = -1 + (a_1 z_1 + 2 a_{11} v_1)\left(\frac{1 - w_1}{2 w_1 N}\right) + (a_{12}^2 z_1 z_2 - b_{12}^2 \bar{z}_1 \bar{z}_2)\left(\frac{1 - 2 w_1}{2 w_1 N}\right) \; ; \tag{79}$$

$$E_{12} = -(a_2 z_2 + 2 a_{22} v_2)\frac{1}{2 N} + (a_{12}^2 z_1 z_2 - b_{12}^2 \bar{z}_1 \bar{z}_2)\left(\frac{1 - 2 w_1}{2 w_1 N}\right) \; ; \text{ and} \tag{80}$$

$$E_{1I} = 1 - (a_1 z_1 + 2 a_{11} v_1)\left(\frac{1 - w_1}{2 w_1 N}\right) + (a_2 z_2 + 2 a_{22} v_2)\frac{1}{2 N} - (a_{12}^2 z_1 z_2 - b_{12}^2 \bar{z}_1 \bar{z}_2)\left(\frac{1 - 2 w_1}{w_1 N}\right). \tag{81}$$

It can be verified that the price and income elasticities above sum to zero. That is, the minflex Laurent demand system satisfies homogeneity.

Combining [79] and [80] with [34] gives the HOPE equation in the minflex Laurent model,

$$s_{11} = -\frac{I}{p_1^2}\left\{(1 - w_1) w_1 - (a_{12}^2 z_1 z_2 - b_{12}^2 \bar{z}_1 \bar{z}_2)\frac{(1 - 2 w_1)^2}{2 N} - (a_1 z_1 + 2 a_{11} v_1)\frac{(1 - w_1)^2}{2 N} - (a_2 z_2 + 2 a_{22} v_2)\frac{w_1^2}{2 N}\right\}. \tag{82}$$

A parallel argument to the evaluation of [68] may be followed in evaluating [82]. In this case where $p_1 = p_2 = I$, the non-negativity of w_1 requires from [76] to [78] that the sign of $(a_1 + a_{11} + a_{12}^2 + b_{12}^2)$ be the same as the sign of $(a_2 + a_{22} + a_{21}^2 + b_{21}^2)$. However, it is possible for some parameter values satisfying this condition that s_{11} in [82] will switch signs as w_1 varies in the interval $1 \geq w_1 \geq 0$ when p_1, p_2 and I change.[19] That is, in the absence of more parameter restrictions, the minflex Laurent is not globally well–behaved.

However, the minflex Laurent can be globally well–behaved by first making it homothetic. For this, it is necessary that $a_1 = a_2 = b_{12}^2 = b_{21}^2 = 0$ so that income, I, can be factored out of the share equation. The results are similar to those for the homothetic generalized Leontief. In fact, the results in [69] to [72] will apply identically to the minflex Laurent simply by replacing γ_{11} by a_{11} and γ_{12} by a_{12}^2. But by the same argument, homotheticity would not be sufficent for HOPE to be UNP. Additionally, it necessary that the cross–price parameter be zero, i. e., $a_{12}^2 = 0$. Thus, the uniformly zero value of s_{11} in [73] as well the results on elasticities in [74] will apply identically to the two models. That is, in a globally well–behaved minflex Laurent characterized by homothetic preferences and zero Hicksian substitutability, the demands for the two–goods are own–price inelastic and these goods are (gross) complements with unitary income elasticities.

IV. Negative Semi–Definiteness of the HSSM With More Than Two Goods

In the two–good model, HOPE being UNP is equivalent to HSSM being GNSD. However, this equivalence does not generalize to cases where there are more than two goods. The reason is that HSSM of order greater than 2 that is symmetric, singular and has non–positive diagonal elements is not necessarily negative semi–definite because the eigenvalues are not necessarily all non–positive. To show this, suppose there are three goods. Hence, it follows from [30] and [31] by similar reasoning that the characteristic equation of HSSM is

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[19] An example similar to that in footnote 15 for the generalized Leontief can be constructed to show that the minflex Laurent is also not globally well–behaved.

$$-\lambda^3 + s_{11}\lambda^2 + (s_{12}^2 + s_{13}^2 + s_{23}^2 - s_{11}s_{22} - s_{11}s_{33} - s_{22}s_{33})\lambda = 0 \ . \qquad [83]$$

From [83], one of the eigenvalues must be zero. However, the other two are not necessarily non–positive. The reason is that the coefficient of λ in the third term could be of any sign, given that s_{11}, s_{22} and s_{33} are all non–positive diagonal elements. While it remains necessary that HOPE is non–positive, this sign condition is not anymore sufficient for HSSM to be negative semi–definite if there are more than two goods.[20]

The difficulty of insuring that HSSM is negative semi–definite could be appreciated once it is realized that analytic solutions to the eigenvalues may not exist for matrices of higher orders. For example, if there are five goods, then the characteristic equation of SM yields a quintic (fifth order) polynomial function of the eigenvalue, λ. But it has been proved by the mathematician Galois that a quintic polynomial has no analytic solution in the sense that there cannot exist a formula for its eigenvalues (Strang, 1980). In this case, the eigenvalues can only be computed numerically.[21]

Moreover, the requirement that HSSM be GNSD is complicated by the fact that each element of the HSSM is a function of parameters and expenditure shares. As illustrated by the HOPE equations in the two–good case, the functional relation could only be more complicated with more goods such that it may not be as straightforward compared to the two–good case to impose parameter restrictions to insure that HSSM is GNSD. Furthermore, the numerical procedures for insuring this result require that the parameters

[20] Note that HOPE is a diagonal element of the HSSM and must, therefore, be non–positive in order for HSSM to be negative semi–definite.

[21] Consider that imposing negative semi-definiteness on a matrix implies imposing restrictions to insure that all eigenvalues are non-positive. Consider also that eigenvalues are the roots of a polynomial function defined by the characteristic equation of a matrix. However, as noted in Strang (1980), there does not exist an analytic solution to the roots of a quintic or fifth order polynomial or of polynomials of higher orders. In such cases, the eigenvalues can only be computed numerically. This is in contrast to the analytical solution of the eigenvalues of a 2 x 2 matrix by the quadratic formula.

can be decoupled from the expenditure shares so that negative semi–definiteness can be imposed on the parameter matrix. Although it is not obvious from the HOPE equations how this could be done for the HSSM , this procedure was applied by Diewert and Wales (1987) to the Hessian matrix (which is conceptually identical to the HSSM) of cost functions. The procedure first requires decomposing the quadratic form of the Hessian (divided by the total cost) into the sum of the matrix of parameters and a negative semi–definite matrix formed from the vector of expenditure shares. Once done, negative semi–definiteness can be imposed numerically on the matrix of parameters based on techniques by Wiley, Schmidt and Bramble (1973) and by Jorgenson and Fraumeni (1981). However, even if possible, this could result in the loss of the flexibility property of the demand system.[22] With this loss, the model would not be capable of achieving the full range of price and income elasticities at any particular data point, as shown by the results in the two–good case.

VI. Conclusion

It is the view of this paper that it may be desirable simply to leave negative semi-definiteness an empirical issue for the following reasons. One is founded on the unrealistic consequences on the price and income elasticities demonstrated by the analytical results in the two-good case. The other is that it gets extremely difficult to impose negative semi-definiteness *a priori* as the number of goods increases and it becomes analytically impossible if there are five or more goods. In the latter case, numerical procedures

[22]For example, Diewert and Wales pointed out that the Jorgenson–Fraumeni procedure destroyed the flexibility property of the translog cost function.

require that the non–negative vector of expenditure shares and the matrix of parameters can be separated so that negative semi–definiteness can be imposed on this matrix. Although this may be possible, chances are that this will impair the flexibility properties of the demand system. The likelihood of this occurring for Marshallian demand systems in the general case is evidenced by the findings of this paper in the case of two goods.

In suggesting that negative semi-definiteness be resolved empirically, this paper nevertheless considers it a minimal property that the estimated model satisfy this property at every data point of the sample of observations. Considering that the negative semi-definiteness of the HSSM is a necessary and sufficient condition for concavity of the expenditure function, then an empirically estimated HSSM that is negative semi-definite at every data point is *prima facie* evidence that the estimated demand model embodies expenditure minimization or, by duality, also utility maximization, given the other global properties of additivity, homogeneity and symmetry. This validates the model for practical applications, for example, in welfare change measurement and in forecasting. It should be obvious that a demand system is invalid as a tool for welfare change measurement unless it embodies utility-maximizing behavior. Moreover, since inferences in econometrics are based on the properties of the model at every data point, the estimated model is not reliable for forecasting future demand predicated on utility-maximizing behavior unless the model captures this behavior at every point in the sample of past observations.

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