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A Money Metric Measure of Welfare Change From Multiple Price and Income Changes

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Abstract

McKenzie's money metric measure of the Hicksian Equivalent Variation (HEV) is derived from a third-order Taylor series approximation to the change in an arbitrary indirect utility function. This paper reformulates the money metric to establish its validity as a welfare change indicator. By means of this reformulation, a third-order money metric is shown to be non-decreasing in income, non-increasing in prices and homogeneous of degree zero in income and prices. These are the essential economic properties of an indirect utility function that a valid welfare change indicator must also possess. The money metric satisfies the integrability conditions but without the restriction of identical income elasticities of demand required for the integrability of the traditional Marshallian Consumer's Surplus (MCS). Welfare changes from multiple price and income changes in a demand system can be measured by the money metric, using no more information than that required to calculate the MCS. The superiority of the reformulation of the third-order money metric over the MCS as an approximation to the HEV is demonstrated in the more general case of unequal income elasticities where the MCS is not integrable or not uniquely measurable. It is argued that the HEV is the only meaningful money measure of welfare change in this general case when there are multiple price and income changes. However, the HEV can only be approximated in practice, there being no exact measure in the absence of knowledge about the exact form of the underlying utility function or the structure of consumer preferences. As a superior approximation to the HEV compared to the MCS, the money metric rests on a sound theoretical basis and has the potential for wide applications in policy analyses, given the fact that it is measurable from the typical Marshallian demand functions estimated in practice.

I. Purposes of this Paper and Summary of Findings

This paper is principally concerned with the measurement of the money equivalent of the change in welfare induced by policies that change consumer prices and income. The underlying premise is that consumers are utility-maximizing and this behavior is exemplified by observable market demand functions.

Policy analyses, in general, often use the Marshallian Consumer's Surplus (MCS) to evaluate alternative scenarios.[1] This paper presents a method of calculating the money equivalent of the change in welfare resulting from multiple price and income changes using a money metric originally proposed by McKenzie as an alternative to the MCS.[2] However, McKenzie's original specification is reformulated in order to show that the money metric has the properties of an indirect utility function, thereby establishing it as a valid indicator of welfare change. In particular, it is shown that the reformulation is non-decreasing in income, non-increasing in prices and homogeneous of degree zero in income and prices.

The money metric is proposed for application in situations where the MCS is not integrable or, equivalently, not uniquely measurable. It is shown that the money metric satisfies the integrability conditions implied by utility maximization without, however, the restrictions on demand functions or on preferences required for the integrability of the MCS. At the same time, the money metric is practical in that it is derived from exactly the same Marshallian demand functions that are estimated to calculate the MCS. Thus, it is more appealing than the MCS in being less restrictive in theory as well as in applications.

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[1]For a textbook discussion of the use of the MCS in agricultural economics, see Just, R. E., Hueth, D. L. and Schmitz, A., Applied Welfare Economics and Public Policy (Englewood Cliffs: Prentice Hall, Inc., 1982) or McCalla, A. F. and Josling, T. E., Agricultural Policies and World Markets (New York: Macmillan Publishing Company, 1985). Moreover, for the use of the MCS in practical applications, see Haley, S. L. and Dixit, P. M., Economic Welfare Analysis: An Application to the SWOPSIM Modeling Framework, Staff Report No. AGES871215 (Washington, DC: U.S. Department of Agriculture, Economic Research Service, August 1988).

[2]McKenzie, G. W., Measuring Economic Welfare: New Methods (Cambridge, England: Cambridge University Press, 1983).

Section 2 outlines the derivation from utility maximization of an integral expression for welfare change in the general case of a simultaneous change in income and prices. This expression serves as the point of departure for both the MCS and the money metric measure.

Section 3 synthesizes the earlier findings in the literature on the conditions for the integrability of the MCS.[3] This paper confirms the postulate that, as a general condition, the MCS is integrable if and only if the income elasticities of demand are equal for the goods whose prices are changing simultaneously.[4]

To accomodate every possible combination of price changes, including the case where all prices change, the above general condition implies that the MCS is integrable if and only if all the income elasticities are equal to one. With all prices changing at the same time, the MCS is integrable if and only if preferences are homothetic since only then will all income elasticities be uniformly unitary. [5] Note, however, that homotheticity is both necessary and sufficient for the integrability of MCS only when all prices change. Otherwise, it is only sufficient and not necessary.[6] In other words, it is in general theoretically correct to use the MCS if the set of changing prices corresponds to the set of commodities with identical

[3]The MCS is mathematically a line integral when there is a change in more than one price at the same time. In this case, the integrability conditions are those required for the MCS to yield a unique value independently of the path of integration.

[4]See Just, R. E., Hueth, D. L. and Schmitz, A.,op. cit., appendix B, pp. 357 – 416, p. 368, in particular.

[5]Silberberg, E., The Structure of Economics: A Mathematical Analysis (New York: McGraw Hill Book Company, 1978), pp. 356 – 59. For an earlier analysis leading to the same conclusion, see his paper, "Duality and the Many Consumer Surpluses," American Economic Review, vol.62, no.5 (December 1972) pp. 942 – 52. For similar conclusions and other related findings in a wider set of cases, see Chipman, J. S. and Moore, J. C., "Compensating Variation, Consumer's Surplus, and Welfare," American Economic Review, vol. 70, no. 5 (December 1980), pp. 933 – 49. Also by Chipman and Moore, see "The Scope of Consumer's Surplus Arguments," in Tang, A. M., et. al., eds., Evolution, Welfare and Time in Economics: Essays in Honor of Nicholas Georgescu-Rogen (Lexington, Mass.: Heath - Lexington Books, 1976), pp. 69 – 123.

[6]It seems that Silberberg, on the one hand, as well as Chipman and Moore, on the other, are analyzing the case where all prices are changing at the same time since they are all emphatic about both the necessity and sufficiency of homothetic preferences. In contrast, Just, Hueth and Schmitz are analyzing the same cases treated here, where a subset of prices change.

income elasticities. However, this situation is unlikely to occur in reality and, therefore, the MCS is unlikely to be reliable in practice.

Moreover, section 3 shows that when all prices but one are changing, the MCS is also integrable if preferences are parallel with respect to the good with the constant price. In this case, the latter good has an income elasticity at least equal to one and all the other goods with varying prices have income elasticities equal to zero. However, parallel preferences constitute a sufficient but not a necessary condition for the integrability of MCS since the goods with varying prices need only have uniform income elasticities, not necessarily equal to zero.

Section 4 supports earlier findings in the literature that in the more general case of unequal income elasticities, where the MCS is not uniquely measurable, the only meaningful money measure of welfare change is the Hicksian Equivalent Variation (HEV).[7] This puts into perspective the potential role of the money metric since it is precisely intended to approximate the HEV from the typical Marshallian demand functions estimated in applications.

Section 4 proceeds to derive the money metric according to the integrability conditions implied by utility maximization for demand systems with unequal income elasticities. Moreover, section 4 explains why the money metric can be properly interpreted as a measure of the HEV although it is derived entirely from the Marshallian demand functions used to calculate the MCS.

Section 5 presents a reformulation of McKenzie's money metric and shows that it has the properties of an indirect utility function, namely, that it is non-decreasing in income, non-increasing in prices and homogeneous of degree zero in income and prices. It is shown that it has these properties if it is a first or third-order approximation but not if it is a second-order approximation. By this reformulation, it is shown that a third-order approximation, in particular, is a theoretically valid indicator of welfare change.

Section 6 presents a simple example where the MCS does not yield a unique

[7]See Chipman, J. and Moore, J. C., op. cit., p. 948; and Hurwicz, L. and Uzawa H., "On the Integrability of Demand Functions," in Chipman, J. S., Hurwicz, L., Richter, M. K., and Sonnenschein, H. F., eds., Preferences, Utility and Demand (New York: harcourt Brace, 1971), pp. 114 – 48.

value and demonstrates, using this example, the relative accuracy of the third-order money metric over the MCS in measuring the HEV.[8] It is argued that the HEV is the best money measure of welfare change for policy analyses. The example shows that the money metric is superior to the MCS for the measurement of welfare changes.

Finally, section 7 concludes this paper with a brief recapitulation of findings and gives some qualifications about using the money metric. The overall conclusion, however, is that it is a much better measure than the MCS.

II. An Integral Measure of Welfare Change From the Indirect Utility Function

A valid integral measure of welfare change can be defined as a discrete change in an indirect utility function resulting from a change in a consumer's price and income vector. This is derived in the following discussion.[9]

Let the utility function defined by

$$U = U(x_1, x_2, ..., x_n) \tag{1}$$

be maximized subject to the budget constraint

$$I = p_1 x_1 + p_2 x_2 + ... + p_n x_n = \sum_i p_i x_i \tag{2}$$

where I represents income, x_i and p_i, $i = 1, 2, ..., n$, represent, respectively, the quantities and prices of the goods. Let the Marshallian demand functions resulting from utility maximization be

$$x_j = x_j(p_1, p_2, ..., p_n; I) \quad ; \quad j = 1, 2, ..., n. \tag{3}$$

[8]This example is from Silberberg, op. cit., p. 355.

[9]It is assumed that the consumer's preferences and budget set have all the properties that guarantee the existence of well-behaved demand functions. For details, see Varian, H. R., Microeconomic Analysis, Second Edition (New York: W. W. Norton and Company, 1984), ch. 3.

The indirect utility function is obtained by substituting (3) into (1). This is given by

$$V(P,I) = U\,[x_1(P,I),\, x_2(P,I),\, ...,\, x_n(P,I)] \tag{4}$$

where $(P,I) = (p_1, p_2, ..., p_n; I)$ is the vector of goods prices and income. Totally differentiating (3) and (4), [10]

$$dV = \sum_i \frac{\partial V}{\partial p_i}\,dp_i + \frac{\partial V}{\partial I}\,dI = \sum_j \frac{\partial U}{\partial x_j}\,dx_j \qquad \textit{and} \tag{5}$$

$$dx_j = \sum_i \frac{\partial x_j}{\partial p_i}\,dp_i + \frac{\partial x_j}{\partial I}\,dI\,. \tag{6}$$

Therefore, by substitution of (6) into (5),

$$dV = \sum_i \sum_j \frac{\partial U}{\partial x_j}\,\frac{\partial x_j}{\partial p_i}\,dp_i + \sum_j \frac{\partial U}{\partial x_j}\,\frac{\partial x_j}{\partial I}\,dI\,. \tag{7}$$

Let λ be the Lagrange multiplier from the maximization of (1) subject to (2). Thus, the first order conditions yield

$$\frac{\partial U}{\partial x_j} = \lambda\, p_j \quad ; \quad j = 1, 2, ..., n \tag{8}$$

so that (7) can be rewritten as

$$dV = \lambda \left[\sum_i \sum_j p_j\,\frac{\partial x_j}{\partial p_i}\,dp_i + \sum_j p_j\,\frac{\partial x_j}{\partial I}\,dI\right]. \tag{9}$$

By partial differentiation of the budget constraint in (1) with respect to p_i and also with respect to I, the results are, respectively,

$$\sum_j p_j\,\frac{\partial x_j}{\partial p_i} = -\,x_i \tag{10}$$

[10]Notice that there are n different goods and, therefore, also n different prices. With this in mind, all summations from hereon should be understood to encompass all goods or prices 1,2, ..., n no matter what the summation index is, for example, i, j, or k depending on the context of the derivation.

and

$$(11) \qquad \sum_j p_j \frac{\partial x_j}{\partial I} = 1 .$$

Substituting (10) and (11) into (9) and recalling (5),

$$(12) \qquad dV = \sum_i \frac{\partial V}{\partial p_i} dp_i + \frac{\partial V}{\partial I} dI = \sum_i (-\lambda x_i) dp_i + \lambda dI .$$

That is,[11]

$$(13) \qquad \frac{\partial V}{\partial p_i} = -\lambda x_i \quad ; \quad \frac{\partial V}{\partial I} = \lambda .$$

Finally, by integrating (12), the discrete change in utility corresponding to a change in a consumer's price-income vector is given by

$$(14) \qquad \Delta V = \int_{p_i^o}^{p_i^n} \sum_i (-\lambda x_i) dp_i + \int_{I^o}^{I^n} \lambda dI$$

where p_i^o , p_i^n , $i = 1, 2, ..., n$ are the elements of the old and new price vectors and I^o and I^n are the old and new income levels, respectively.[12]

Equation (14) is the integral expression for welfare change that forms the basis of the MCS and money metric measures. The basic point of departure between these two measures crucially depends on the assumptions about the functional behavior of the marginal utility of income, λ. These assumptions permit the conversion of ΔV into a money equivalent measure of welfare change, at the same time that they imply restrictions on the demand functions or on the underlying preferences. The implications of these assumptions for the MCS are discussed in the next section.

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[11] Ibid., p. 126. By Roy's identity, the Marshallian demand function may be obtained as the negative of the ratio of these partial derivatives, i.e., $-(\partial V/\partial p_i) / (\partial V/\partial I) = x_i$.

[12] For a result similar to (14), see Just, R. E., Hueth, D. L. and Schmitz, A., op. cit.

III. Conditions for the Integrability of MCS in Terms of Restrictions on Demand Functions or Preferences

The MCS is a money measure of welfare change given a change in prices, usually with no contemporaneous change in income. In the more general case where more than one price changes at the same time, the MCS is derived from (14) by factoring out λ from the line integral expression in the right-hand side and then dividing it into ΔV. Given the same income or $\Delta I = 0$, we obtain (15) which shows MCS as a line integral in that it is a sum of integrals, each corresponding to the change in welfare due to a change in a specific price when there is a simultaneous change in prices.[13]

$$MCS = \frac{\Delta V}{\lambda} = \int_{p_i^o}^{p_i^n} \sum_i (-x_i)\, dp_i\,. \tag{15}$$

Notice that MCS is in money terms. This is because ΔV is in units of utility and $1/\lambda$ is the marginal cost (in terms of income) of utility, λ being the marginal utility of income.

The derivation of MCS above is not innocuous as it appears because factoring out λ from the line integral is legitimate if and only if λ is constant with respect to each of the prices that have changed. However, this condition is generally not true. While there may be no change or only a once and for all change in λ due to a change in income alone, the value of λ could change from one integral term to the other as price changes. If so, it cannot be factored out of the line integral and then be divided into ΔV in order to obtain the formula for MCS in (15).

[13] In order for this line integral to yield a unique value, it has to be independent of the path of integration. This path independence is precisely the integrability condition. Thus, if a line integral is path independent, then it is integrable in the sense that its value is unique. See Danese, Arthur E., Advanced Calculus (Boston: Allyn and Bacon, Inc., 1965), pp. 123 – 29. This integrability issue does not, however, arise in the case of a single price change. If only the kth price changes, then $MCS = -\int_{p_k^o}^{p_k^n} x_k\, dp_k$ which is a simple integral and always integrable. This is familiar in textbooks as the area under the Marshallian demand curve.

In this analysis, the functional behavior of λ with respect to the prices that have not changed is irrelevant since these latter prices are not represented in the line integral expression. Hence, in order to factor out λ from the line integral, it is necessary and sufficient that it be constant with respect to each of the changing prices. In this case, MCS is integrable and the integrability condition for (15) is that

$$\frac{\partial \lambda}{\partial p_i} = 0 \tag{16}$$

for all prices that have changed. Alternatively, the integrability condition may be given as[14]

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial x_j}{\partial p_i} \quad , \quad i \neq j \tag{17}$$

for the same set of changing prices. Conditions (16) and (17) are equivalent, i.e.,

$$\left[\frac{\partial \lambda}{\partial p_i} = \frac{\partial \lambda}{\partial p_j} = 0\right] \Leftrightarrow \left[\frac{\partial x_i}{\partial p_j} = \frac{\partial x_j}{\partial p_i}\right] \quad . \tag{18}$$

This means that the constancy of λ with respect to prices is equivalent to the symmetry of the Marshallian cross-price effects between the corresponding goods. This condition for the integrability of MCS implies, however, a restriction on demand functions because the symmetry of Marshallian cross-price effects is not a necessary result from consumer theory.[15]

Suppose that (16) is true for all prices that changed. To see its implication, consider again (13). This yields by the chain rule,

$$\frac{\partial \lambda}{\partial p_i} = \frac{\partial\left(\frac{\partial V}{\partial I}\right)}{\partial p_i} = \frac{\partial\left(\frac{\partial V}{\partial p_i}\right)}{\partial I} = -\lambda \frac{\partial x_i}{\partial I} - x_i \frac{\partial \lambda}{\partial I} \qquad \textit{and} \tag{19}$$

$$\frac{\partial \lambda}{\partial p_j} = \frac{\partial\left(\frac{\partial V}{\partial I}\right)}{\partial p_j} = \frac{\partial\left(\frac{\partial V}{\partial p_j}\right)}{\partial I} = -\lambda \frac{\partial x_j}{\partial I} - x_j \frac{\partial \lambda}{\partial I} \quad . \tag{20}$$

[14]See Just, R. E., Hueth, D. L. and Schmitz, A., op. cit., p. 364 and also Silberberg, E., The Structure of Economics: A Mathematical Analysis (New York: McGraw-Hill Book Company, 1978), p. 356.

[15]This is in contrast to the symmetry of Hicksian cross-price effects which is necessarily true.

If (16) is true, it then follows from (19) and (20) that

$$(21) \qquad \frac{\partial x_i}{\partial I}\frac{I}{x_i} = \frac{\partial x_j}{\partial I}\frac{I}{x_j} = -\frac{\partial \lambda}{\partial I}\frac{I}{\lambda} \quad .$$

Thus, (16) implies (21) but the latter is true only if the former is true. Therefore, (21) means that the income elasticities of demand are equal for the goods whose prices have changed if and only if λ is constant with respect to each of these prices.[16] If so, λ can be factored out of the line integral in (14) to derive MCS in (15). In this case, MCS is integrable or uniquely measurable.

Suppose now that (17) is true. In order to prove the equivalence in (18), consider the Slutsky equations

$$(22) \qquad \frac{\partial x_i}{\partial p_j} = \frac{\partial x_i^h}{\partial p_j} - x_j\frac{\partial x_i}{\partial I} \qquad \textit{or}$$

$$(23) \qquad \frac{\partial x_j}{\partial p_i} = \frac{\partial x_j^h}{\partial p_i} - x_i\frac{\partial x_j}{\partial I} \quad .$$

Notice that x_i and x_j are Marshallian, in contrast to x_i^h and x_j^h which are Hicksian demand functions. By using the fact that the Hicksian cross-price effects are always symmetric, i.e.,[17]

$$(24) \qquad \frac{\partial x_i^h}{\partial p_j} = \frac{\partial x_j^h}{\partial p_i}$$

it follows from (22), (23) and (24) that, if (17) is true, then

$$(25) \qquad x_j\frac{\partial x_i}{\partial I} = x_i\frac{\partial x_j}{\partial I} \quad .$$

[16]This result may be found in Just, R. E., et. al. op. cit., p. 368.

[17]The symmetry of Hicksian cross-price effects is easily proved. By Shephard's lemma, the Hicksian demand function is the first-order partial derivative of the expenditure function with respect to price. Now, the Hicksian cross-price effect is the first-order cross-price partial derivative of the Hicksian demand function. But following the application of Shephard's lemma, this Hicksian cross-price effect is equal to the second-order cross-price partial derivative of the expenditure function. But for any two distinct prices, these second-order cross-price partial derivatives are equal by Young's theorem. This establishes the equality or symmetry of Hicksian cross-price effects. See Varian, H. op. cit., p. 133.

Finally, by multiplying both sides of (25) by $\frac{I}{x_i x_j}$, the result is that

$$\frac{\partial x_i}{\partial I}\frac{I}{x_i} = \frac{\partial x_j}{\partial I}\frac{I}{x_j} \tag{26}$$

which was obtained earlier in (21). As shown above, (17) implies (26). Observe, however, that (26) is true only if (17) is true. This means that the symmetry of Marshallian cross-price effects is necessary and sufficient for the equality of income elasticities of demand.

It follows from above that the constancy of λ with respect to each of the prices that are changing and the symmetry of the Marshallian cross-price effects between the corresponding goods are equivalent integrability conditions. By way of the Slutsky equation, this implies that the equality of income elasticities of demand is a necessary and sufficient condition for the integrability of MCS. This equality says nothing specific about the value of the income elasticities, as can be seen from (21). This result holds so long as only a subset of prices change.

However, if all prices change at the same time, the above result implies that the MCS is integrable if and only if all income elasticities are unitary, i.e., preferences are homothetic. To demonstrate this, consider first the fact that λ is the partial derivative of the indirect utility function, $V(P,I)$, with respect to income, I. Hence, it is also a function of the price vector, P, and of I. Since $V(P,I)$ is homogeneous of degree zero in (P,I), then λ is homogeneous of degree minus one in (P,I). Therefore, [18]

$$\lambda = \frac{\partial V}{\partial I} = -\sum_i \frac{\partial \lambda}{\partial p_i} p_i - \frac{\partial \lambda}{\partial I} I \tag{27}$$

by Euler's theorem.[19] Moreover, because the budget constraint is assumed binding, the Lagrange multiplier, λ, is non-zero. In fact, it must be positive since it is also

[18] This follows from the fact that the partial derivative of a function which is homogeneous of degree k is homogeneous of degree $k-1$. See Henderson, J. M. and Quandt, R., Microeconomic Theory: A Mathematical Approach, Second Edition (New York: McGraw-Hill Book Company, 1971), p. 79. $V(P,I)$ is homogeneous of degree zero in prices and income since it is obtained by substitution into the original utility function of the Marshallian demand functions, which have zero degree homogeneity with respect to prices and income. That is, the quantities demanded are invariant with respect to proportionate changes in prices and income.

[19] For a discussion of Euler's theorem on homogeneous functions, refer to Chiang, A. C.,

the marginal utility of income. Therefore, in view of (27), it cannot be constant both with respect to all prices and income.[20] That is, λ can be constant with respect to all prices alone; with respect to income alone or; with respect to some, but not all, prices and income together.

But suppose now that λ is constant with respect to all prices, i.e,

$$\frac{\partial \lambda}{\partial p_i} = 0 \quad ; \quad i = 1, 2, ..., n. \tag{28}$$

If so, it follows from (19),(20), (27) and (28) that

$$\lambda = -\frac{\partial \lambda}{\partial I} I \quad \textit{and} \tag{29}$$

$$\lambda \frac{\partial x_i}{\partial I} = -x_i \frac{\partial \lambda}{\partial I} \quad ; \quad i = 1, 2, ..., n. \tag{30}$$

Given that $\frac{\partial \lambda}{\partial I} \neq 0$, these last two results yield

$$\frac{\partial x_i}{\partial I} \frac{I}{x_i} = 1 \quad ; \quad i = 1, 2, ..., n. \tag{31}$$

This result can also be obtained by substituting the value of λ in (29) into the right-hand side of (21). The result in (31) that all income elasticities of demand are uniformly unitary is equivalent to the homotheticity of preferences.[21] That is, in the case where all prices change, the MCS is integrable if and only if preferences are homothetic.

The reason for the necessity of homothetic preferences when all prices are changing is that it is necessary that all income elasticities be equal according to (21). However, if all income elasticities are equal, they must all be equal to one, from the fact that the sum of income elasticities weighted by budget shares equals

Fundamental Methods of Mathematical Economics, Third Edition (New York: McGraw-Hill Book Company, 1984), pp. 410 – 17.

[20] Although arguing by means of a different set of equations, this same assertion was earlier made by Samuelson, P. A., Foundations of Economic Analysis, Enlarged Edition (Cambridge: Harvard University Press, 1983), p. 191.

[21] See Silberberg, op. cit., pp. 356 – 60.

one and that the sum of these shares equals one.[22] But when income elasticities are all equal to one, preferences are homothetic, by definition. Sufficiency follows from the mere fact that in this case all income elasticities are uniform, therefore, consistent with the equality condition in (21).

Finally, the case of parallel preferences can be derived as a special case of the preceding analysis. Parallel preferences imply that income elasticities are zero for some goods. At the extreme, if there are n goods, at most $n-1$ can have zero income elasticities. The reason for this upper limit on the number of possible zero income elasticities follows from the budget constraint. Recalling (11), we obtain without losing generality by changing the summation index j to i,

$$\sum_i \frac{p_i x_i}{I} \frac{\partial x_i}{\partial I} \frac{I}{x_i} = 1 \tag{32}$$

which means that the sum of all income elasticities, weighted by the budget shares, equals one. Clearly, at most $n-1$ of these income elasticities could be zero without violating the budget constraint. Now, if income elasticities are not only equal but also zero, then (21) implies that

$$\frac{\partial \lambda}{\partial I} = 0 \quad . \tag{33}$$

Given (33), it follows in turn from (27) that

$$\lambda = -\sum_i \frac{\partial \lambda}{\partial p_i} p_i \quad . \tag{34}$$

Since λ cannot be zero, at most $n-1$ of this sum of partial derivatives is zero. Suppose that the only non-zero partial derivative is for the kth good. Then, from (19) or (20)

$$\frac{\partial \lambda}{\partial p_k} = -\lambda \frac{\partial x_k}{\partial I} \tag{35}$$

since (33) is assumed in this case. Thus, by substituting (35) into (34), it follows that

$$p_k \frac{\partial x_k}{\partial I} = \frac{p_k x_k}{I} \frac{\partial x_k}{\partial I} \frac{I}{x_k} = 1 \tag{36}$$

[22] This may be seen later in equation (32).

which can also be obtained from the budget constraint relation in (32) if the kth good is the only good with a non-zero income elasticity. It can be deduced that this good must have an income elasticity at least equal to one from the fact that its share in the budget is at most equal to one. If so, all the income elasticities are uniformly zero for all the other goods whose prices are free to vary. Thus, this uniformity implies that parallel preferences suffice for the integrability of MCS. However, the case of parallel preferences while sufficient is not necessary. The reason is that the necessary and sufficient condition is for all income elasticities, except one, to be equal. But the budget constraint relation in (36) implies that it is possible for all income elasticities, except one, to be uniformly equal to some number other than zero. Thus, the case of parallel preferences is not necessary.

To summarize, if only a subset of prices change simultaneously, it is necessary and sufficient for the integrability of MCS that the income elasticities of the goods with changing prices be uniformly equal to some value, but not necessarily equal to one or zero. This uniformity does not require homothetic or parallel preferences but is satisfied given these preferences. [23] That is, homothetic or parallel preferences constitute sufficient but not necessary conditions for the integrability of MCS. Since these preferences guarantee uniformity of income elasticities (all equal to one, if homothetic, or equal to zero for all but one good, if parallel), it follows that, given either type of preferences, MCS is integrable irrespective of the prices that are changing, except for the price of the one good with a non-zero income elasticity under parallel preferences. Unfortunately for the MCS, it also follows that it is not integrable in the more general case of unequal income elasticities where more than one price varies at the same time.

IV. A Money Metric Measure of Welfare Change for the General Case of Unequal Income Elasticities

The integrability problem surrounding the MCS has led Silberberg to say that: "The simple truth is that there is no unique dollar or money equivalent of a

[23]This is the same conclusion arrived at by Just, R. E., et. al, in our earlier reference.

change in utility. There is nothing to do about it."[24]. This remark goes too far, however, for it confuses the means of measurement and what is being measured. The problem with MCS is that, in general, it does not yield a unique value. This does not, however, prove that the welfare change from these changing prices is not measurable. On the contrary, as noted by Chipman and Moore, the theoretical possibility of computing a function, which is a money measure of indirect utility, is not in question since the existence of this function has been proved earlier by Hurwicz and Uzawa.[25] They conclude that all that is needed is a practical algorithm to compute the Hurwicz-Uzawa income-compensation function, which in the view of Chipman and Moore is the same as the generalized Hicksian Equivalent Variation (HEV).

The suggestion of Chipman and Moore to compute the HEV is for the general case, outside of homothetic or parallel preferences. The HEV is preferred to the other Hicksian measure of welfare change, the Hicksian Compensating Variation (HCV), for policy analyses because the HEV is the natural procedure for comparing a base case with a series of alternative scenarios. By means of the HEV, alternatives are evaluated by comparing the income equivalent (using the "old" set of prices as the base) of a change in utility induced by the effect of policies on prices and income.

Chipman and Moore, as well as Silberberg, argue that the HCV has additional problems as an indicator of welfare change. The issue is not integrability since it is always integrable, like HEV, considering that its integrability only requires the symmetry of Hicksian cross-price effects, which is always true. The problem arises from the computation of HCV in that the income compensation from the price change is based on the "new" vector of prices but holding utility constant at the "old" level.[26] This being the case, converting HCV into an ordinal indi-

[24] Silberberg, op. cit., p. 353

[25] See Chipman, J. and Moore, J. C., op. cit., p. 948; and Hurwicz, L. and Uzawa H., "On the Integrability of Demand Functions," in Chipman, J. S., Hurwicz, L., Richter, M. K. and Sonnenschein, H. F., eds., Preferences, Utility and Demand (New York: Harcourt Brace, 1971), pp. 114 – 48.

[26] For example, suppose that prices fall. In this case, the HCV is the maximum amount of money that can be taken away from the consumer at the same time leaving him as well off as

cator of welfare change is problematic in view of the fact that it is computed by holding utility constant, i.e., no change in welfare. However, Chipman and Moore showed that the HCV is an acceptable indicator of welfare change if and only if the underlying preferences are homothetic or parallel.

In turn, the above finding implies that attempts to approximate the MCS by the HCV, as proposed by Willig, is either futile or unnecessary.[27] The reason for futility is that if preferences are not homothetic, the HCV is not an acceptable indicator of welfare change at the same time that the MCS is not integrable, i.e., not uniquely measurable. However, if preferences are homothetic, the HCV is acceptable but the MCS is integrable so that an approximation is unnecessary. [28]

It appears that in the general case where the income elasticities vary among goods, i.e., outside of homothetic or parallel preferences, the only meaningful measure of welfare change is the HEV. The reason is that the HEV can properly be interpreted as the money equivalent of welfare change. It is computed based on the old vector of prices but allows a change in utility by holding it constant at the new level. Thus, suppose prices fall so that welfare improves or utility increases to a higher level. Holding utility constant at this new (higher) level, the HEV is computed based on the old prices precisely to determine the amount of money that the consumer will accept to keep him at his new (higher) level of utility if he were to pay the old (higher) prices. That is, the consumer should be indifferent between receiving the full amount of the HEV and paying the old (higher) prices, or simply paying the new (lower) prices. In this case of a fall in prices, the HEV is positive

before the fall in prices. For further discussion of Hicksian compensation measures, see Boadway, R. and Bruce N., Welfare Economics (Oxford: Basil Blackwell Ltd., 1984), pp. 39 – 42) and, of course, Hicks, J. R., Value and Capital (London, England: Oxford University Press, 1939; "Consumers' Surplus and Index Numbers,"Review of Economic Studies, 9 (Summer 1942), pp. 126 – 37; and A Revision of Demand Theory (Oxford, England: Clarendon Press, 1956).

[27]Willig, R. D., "Consumer's Surplus Without Apology," American Economic Review, vol. 66, no.4 (September 1976), pp. 589 – 97.

[28]Chipman and Moore, p. 945, support this conclusion. On the other hand, in his general discussion relating to the HCV and MCS when preferences are not homothetic, pp. 489 – 494, Silberberg concluded, p. 494, that "attempts to use consumer's surplus to measure welfare losses are largely the application of the inappropriate to measure the undefinable." He did not, however, address the issue of measuring welfare changes by the HEV.

and it can be interpreted as the equivalent minimum income subsidy. With a rise in prices, the HEV is negative and it can be interpreted as the maximum income tax that the consumer will be willing to pay in lieu of paying the higher prices.

In light of the foregoing discussion, the money metric may now be derived from the Marshallian demand functions and shown that, indeed, it is a measure of the HEV. Recall at this point the welfare change integral given earlier by (14). Since it will be assumed that income elasticities are not the same for all goods, λ will be treated as a variable with respect to both prices and income. Thus, λ cannot be factored out of the right-hand side of (14) in the manner of the MCS. In this case, the integrability conditions for (14) are that [29]

$$(37)\qquad \frac{\partial(-\lambda x_i)}{\partial p_j} = \frac{\partial(-\lambda x_j)}{\partial p_i} \; ; \qquad \frac{\partial(-\lambda x_i)}{\partial I} = \frac{\partial \lambda}{\partial p_i} \; ; \; and \qquad \frac{\partial(-\lambda x_j)}{\partial I} = \frac{\partial \lambda}{\partial p_j} \; .$$

It is also assumed that the indirect utility function V(P,I) is an analytic function so that ΔV can be computed by a Taylor series expansion which is evaluated subject to the above integrability conditions. The money metric measure proposed by McKenzie is derived from a third-order Taylor series approximation to the change in the indirect utility function. [30]

Recall the first-order total differential of the indirect utility function from (12), which is reproduced below,

$$dV = \sum_i \frac{\partial V}{\partial p_i} dp_i + \frac{\partial V}{\partial I} dI = \sum_i (-\lambda x_i) dp_i + \lambda dI \; .$$

It follows from this that the second and third-order total differentials are

$$(38)\quad d^2V = \sum_i \sum_j \frac{\partial^2 V}{\partial p_i \partial p_j} dp_i\, dp_j + 2\sum_i \frac{\partial^2 V}{\partial p_i \partial I} dp_i\, dI + \frac{\partial^2 V}{\partial I^2} (dI)^2 \qquad and$$

[29] See McKenzie, op. cit., pp. 27.

[30] Ibid., pp. 45 – 6, provides a justification. Furthermore, however, it will be shown in the following section of this paper that while a second-order approximation is not a valid measure of welfare change, a third-order approximation is valid.

$$(39)\quad d^3V = \sum_i \sum_j \sum_k \frac{\partial^3 V}{\partial p_i \partial p_j \partial p_k} dp_i\, dp_j\, dp_k + 3 \sum_i \sum_k \frac{\partial^3 V}{\partial p_i \partial p_k \partial I} dp_i dp_k\, dI$$

$$+ 3 \sum_i \frac{\partial^3 V}{\partial p_i \partial I^2} dp_i\, (dI)^2 + \frac{\partial^3 V}{\partial I^3} (dI)^3 \quad .$$

By definition of a Taylor series expansion of ΔV, [31]

$$(40)\qquad \Delta V = dV + \frac{1}{2!} d^2V + \frac{1}{3!} d^3V + \ldots + \frac{1}{n!} d^nV + R_n \; .$$

Each term corresponding to the nth order total differential is divided by the factorial of n. Moreover, each partial derivative is evaluated at the initial values of the variables and the changes in these variables are defined as discrete changes from initial values. The term R_n is the remainder from a finite nth order approximation.

All the partial derivative terms in the first, second and third-order total differentials are obtained from the earlier results in (13) that

$$\frac{\partial V}{\partial p_i} = -\lambda x_i \qquad ; \qquad \frac{\partial V}{\partial I} = \lambda$$

subject to the integrability conditions in (37). It should be emphasized that λ is treated as a variable with respect to all prices and income in the following derivations.

The crucial steps in deriving the money metric is the normalization of the value of λ at the initial values of prices and income to one, i.e.,[32]

$$(41)\qquad \lambda = \lambda(P^o, I^o) = 1 \; .$$

and the linearization of V at that point with respect to income. This implies that the partial derivatives of λ with respect to income of all orders are identically zero, i.e.,

$$(42)\qquad \frac{\partial^r \lambda(P^o, I^o)}{\partial I^r} = 0 \qquad ; \qquad r \geq 1 \; .$$

[31] See Chiang, A., op. cit., pp. 256–60 and Apostol, T. M., Calculus, Volume I, Second Edition (New York: John Wiley and Sons, Inc., 1967), pp. 278 – 9.

[32] See McKenzie, op. cit., pp. 46 for further explanation of (41) and (42).

Note that (42) is not equivalent to assuming that λ is constant with respect to income because (42) is invoked only at the initial price and income vector. In general, λ is treated as a variable with respect to both prices and income, which is exactly how the partial derivatives in the Taylor series approximation are derived from the start. However, at the end, (41) and (42) are invoked by the money metric method when the partial derivatives are evaluated at initial values of prices and income.

It is interesting to note that the normalized initial value of λ equal to one in (41) is what gives the "money metric" its name. The reason is that, by following the above procedure, it can be verified that after substitution of all the derived expressions for the partial derivatives in (12), (38) and (39) into (40), λ can in fact be factored out of the right-hand side of (40) and be divided into ΔV. But because these are evaluated at initial values, it follows that

$$\frac{\Delta V}{\lambda(P^o, I^o)} = \Delta V \quad .$$

Considering that, by definition, λ is the marginal utility of money then its reciprocal, $\frac{1}{\lambda}$, is the marginal cost of utility. By virtue of (41), this marginal cost of utility is also equal to one at the initial values of income and prices. This implies that a unit of utility equals a unit of money. Therefore, ΔV is equivalently measured in terms of money, i.e., ΔV is a "money metric" measure of utility.

Henceforth, the money metric third-order approximation will be denoted by

$$\frac{\Delta V_3}{\lambda(P^o, I^o)} = \Delta V_3$$

where, by definition,

$$\Delta V_3 \equiv MM_3 \quad .$$

Following the above discussion and using this definition, it may now be shown that McKenzie's money metric measure of welfare change from a simultaneous change in prices and income is given by[33]

$$(43) \qquad MM_3 = \Delta I - \sum_i x_i \,\Delta p_i - \sum_i \frac{\partial x_i}{\partial I}\,\Delta p_i\,\Delta I$$

$$+ (1/2)\sum_i \sum_j \left[x_i \frac{\partial x_j}{\partial I} - \frac{\partial x_i}{\partial p_j} \right] \Delta p_i\,\Delta p_j$$

$$+ (1/6)\sum_i \sum_j \sum_k \left[x_i \frac{\partial^2 x_j}{\partial I \partial p_k} + \frac{\partial x_j}{\partial I}\frac{\partial x_i}{\partial p_k} \right] \Delta p_i\,\Delta p_j\,\Delta p_k$$

$$- (1/6)\sum_i \sum_j \sum_k \left[\frac{\partial^2 x_i}{\partial p_j \partial p_k} + x_i\, x_j \frac{\partial^2 x_k}{\partial I^2} \right] \Delta p_i\,\Delta p_j\,\Delta p_k$$

$$- (1/6)\sum_i \sum_j \sum_k \left[x_i \frac{\partial x_j}{\partial I} - \frac{\partial x_i}{\partial p_j} \right] \frac{\partial x_k}{\partial I}\,\Delta p_i\,\Delta p_j\,\Delta p_k$$

$$+ (1/2)\sum_i \sum_j \left[x_i \frac{\partial^2 x_k}{\partial I^2} + \frac{\partial x_i}{\partial I}\frac{\partial x_k}{\partial I} - \frac{\partial^2 x_i}{\partial p_k\, \partial I} \right] \Delta p_i\,\Delta p_k\,\Delta I$$

$$- (1/2)\sum_i \frac{\partial^2 x_i}{\partial I^2}\,\Delta p_i\,(\Delta I)^2 \ .$$

By the manner in which it was derived, this money metric measure satisfies the integrability conditions of the welfare change integral in (14). However, in contrast to the MCS, the integrability of the money metric does not require uniform income elasticities. This follows from the money metric assumption in that $\frac{\partial \lambda}{\partial p_i} \neq 0$ and that, in general, $\frac{\partial \lambda}{\partial I} \neq 0$, either one of which necessarily implies that income elasticities cannot be uniform for all goods.

It may now be shown that MM_3 is an approximate measure of HEV. The reason for this is inherent in the definition of a Taylor series expansion from which it was derived. Suppose there is no change in income but there is a fall in prices. In this case, the discrete change in prices is naturally defined as deviations from

[33]See also McKenzie, G. W. op. cit, pp. 45–48. Although he did not write out his money metric in full, his expressions from (3.18) to (3.32) may be substituted into (3.15) to yield our result in (43). Note, however, our differences in notation. For example, in contrast to McKenzie's ΔM , we use MM_3 for the discrete change in the indirect utility function that is being approximated by the third-order Taylor series expansion.

the old prices. However, following the Taylor series formula, the partial derivatives with respect to this fall in prices are evaluated based on the old prices. Proceeding in this manner, MM_3 is computed by (43). Prices having fallen, welfare must improve so that MM_3 should be positive since it is the increase in utility in money metric terms. Compare this now with the computation of HEV supposing the same situation of no change in income but there is a fall in prices. In this case, utility increases to a higher level. Holding utility at this higher level, the HEV is the additional amount of money (over the old income) needed to stay at the new level of utility computed at the old prices. Since welfare has improved from this fall in prices, HEV must be positive in order for the consumer to pay the old higher prices. The case of a rise in prices is symmetrical and, by similar reasoning, both MM_3 and HEV must be negative as a result.

V. Properties of the Money Metric as a Welfare Change Indicator

A valid integral measure of welfare change must possess the properties of the indirect utility function from which it is derived. Therefore, it is necessary to demonstrate that the money metric possesses these properties. On this thought, it is rather surprising that McKenzie, himself, who originated the money metric did not formally demonstrate the following properties.

For one thing, a welfare change indicator must be non-decreasing in income. That is, if all that happens is that the consumer's income increases (decreases), his level of welfare must not deteriorate (improve). This means that, if all prices remain constant,

$$(44) \qquad MM_3 \geq 0 \quad if \quad \Delta I \geq 0 \quad or \quad MM_3 \leq 0 \quad if \quad \Delta I \leq 0 \ .$$

Also, it must be non-increasing in prices. That is, if all price changes are negative (positive) with income remaining the same, the level of welfare must not deteriorate (improve). That is,

$$(45) \qquad MM_3 \geq 0 \quad if \quad \Delta p_i \leq 0 \quad or \quad MM_3 \leq 0 \quad if \quad \Delta p_i \geq 0$$

for <u>all</u> $i = 1, 2, ..., n$. Moreover, it must be homogeneous of degree zero in prices and income. This means that the level of welfare must remain the same if all prices and income change in the same proportion. That is,

$$(46) \qquad MM_3 = 0 \quad if \quad \tau = \frac{\Delta I}{I} = \frac{\Delta p_i}{p_i} \quad ; \quad i = 1, 2, ..., n$$

for <u>all</u> values of τ . Among the properties of an indirect utility function, the above appear to have the most straightforward economic interpretation.[34] In principle, these may be considered minimal properties of a welfare change indicator.

In order to demonstrate the above properties, it will be convenient to convert the discrete changes in prices and income into changes as a proportion of their initial values. This will enable a reformulation of the money metric as a proportion of initial income. This last result makes sense in that the money metric, as seen in the preceding section, measures the change in welfare in the same units as income so that MM_3 can be divided by I. Moreover, this reformulation has the added appeal of being unit free since the change in each variable is expressed as a proportion of its initial value. Now, let the budget shares be defined by

$$(47) \qquad \omega_i = \frac{p_i\, x_i}{I} \quad ; \quad \sum_i \omega_i = 1$$

from the income constraint. Without loss of generality, (47) can be rewritten with other subscripts j or k. In view of (47), (43) can be rewritten as

$$(48) \qquad \frac{MM_3}{I} = \frac{\Delta I}{I} - \sum_i \omega_i \frac{\Delta p_i}{p_i} - \sum_i \omega_i \frac{I}{x_i} \frac{\partial x_i}{\partial I} \frac{\Delta p_i}{p_i} \frac{\Delta I}{I}$$

$$+ \frac{1}{2} \sum_i \sum_j \left\{ \omega_i \omega_j \frac{I}{x_j} \frac{\partial x_j}{\partial I} - \omega_i \frac{p_j}{x_i} \frac{\partial x_i}{\partial p_j} \right\} \frac{\Delta p_i}{p_i} \frac{\Delta p_j}{p_j}$$

[34] There are other properties of an indirect utility function that are more mathematical in nature, e. g., continuity with respect to prices and income and quasi-convexity with respect to prices. See Varian, H. R.,<u>op. cit.</u> p. 121.

$$+\frac{1}{6}\sum_i\sum_j\sum_k\left\{\omega_i\,p_j\,p_k\,\frac{\partial^2 x_j}{\partial I\partial p_k}\;+\;\omega_i\,\omega_j\,\frac{p_k}{x_i}\,\frac{\partial x_i}{\partial p_k}\,\frac{I}{x_j}\,\frac{\partial x_j}{\partial I}\right\}\frac{\Delta p_i}{p_i}\,\frac{\Delta p_j}{p_j}\,\frac{\Delta p_k}{p_k}$$

$$-\frac{1}{6}\sum_i\sum_j\sum_k\left\{\frac{p_i\,p_j\,p_k}{I}\,\frac{\partial^2 x_i}{\partial p_j\partial p_k}\;+\;I\,\omega_i\,\omega_j\,p_k\,\frac{\partial^2 x_k}{\partial I^2}\right\}\frac{\Delta p_i}{p_i}\,\frac{\Delta p_j}{p_j}\,\frac{\Delta p_k}{p_k}$$

$$-\frac{1}{6}\sum_i\sum_j\sum_k\left\{\omega_i\,\omega_j\,\omega_k\,\frac{I}{x_j}\,\frac{\partial x_j}{\partial I}\,\frac{I}{x_k}\,\frac{\partial x_k}{\partial I}\;-\;\omega_i\,\omega_k\,\frac{p_j}{x_i}\,\frac{\partial x_i}{\partial p_j}\,\frac{I}{x_k}\,\frac{\partial x_k}{\partial I}\right\}\frac{\Delta p_i}{p_i}\,\frac{\Delta p_j}{p_j}\,\frac{\Delta p_k}{p_k}$$

$$+\frac{1}{2}\sum_i\sum_k\left\{I\,\omega_i\,p_k\,\frac{\partial^2 x_k}{\partial I^2}\;+\;\omega_i\,\omega_k\,\frac{I}{x_i}\,\frac{\partial x_i}{\partial I}\,\frac{I}{x_k}\,\frac{\partial x_k}{\partial I}\;-\;p_i\,p_k\,\frac{\partial^2 x_i}{\partial p_k\partial I}\right\}\frac{\Delta p_i}{p_i}\,\frac{\Delta p_k}{p_k}\,\frac{\Delta I}{I}$$

$$-\frac{1}{2}\sum_i I\,p_i\,\frac{\partial^2 x_i}{\partial I^2}\,\frac{\Delta p_i}{p_i}\left(\frac{\Delta I}{I}\right)^2\;.$$

It is obvious that the property of being non-decreasing in income in (44) is satisfied by (48) when all prices are constant, and the percent change in welfare in money metric terms is, in this case, exactly equal to the percent change in income, i.e.

$$(49)\qquad \frac{MM_3}{I}=\frac{\Delta I}{I}\qquad if\qquad \frac{\Delta p_i}{p_i}=0\,,\qquad i=1,2,...,n.$$

In order to show the properties in (45) and (46), note the following identities implied by the budget constraint:

$$(50)\qquad \sum_j p_j\,\frac{\partial x_j}{\partial p_i}=-\,x_i\quad;$$

$$\sum_i\sum_j p_i\,p_j\,\frac{\partial x_j}{\partial p_i}=-\sum_i p_i\,x_i=-\,I\quad;$$

$$\sum_i\sum_j\frac{p_i\,p_j}{I}\,\frac{\partial x_j}{\partial p_i}=\sum_i\sum_j\omega_j\,\frac{p_i}{x_j}\,\frac{\partial x_j}{\partial p_i}=-\,1\quad;$$

$$\sum_i\sum_j p_i\,p_j\,\frac{\partial^2 x_j}{\partial p_i\partial I}=-\,1\quad;$$

$$\sum_i\sum_j\sum_k\frac{p_i\,p_j\,p_k}{I}\,\frac{\partial^2 x_j}{\partial p_i\partial p_k}=2\quad;$$

$$\sum_i p_i \frac{\partial x_i}{\partial I} = 1 \quad ;$$

$$\sum_i \frac{p_i x_i}{I} \frac{I}{x_i} \frac{\partial x_i}{\partial I} = \sum_i \omega_i \frac{I}{x_i} \frac{\partial x_i}{\partial I} = 1 \quad ; and$$

$$\sum_i p_i \frac{\partial^2 x_i}{\partial I^2} = 0 \quad .$$

Consider the case where income remains the same but all prices change in the same proportion given by

$$(51) \qquad \rho = \frac{\Delta p_i}{p_i} \qquad i = 1, 2, ..., n.$$

Substituting (51) into (48) and making use of the budget identities in (50), it can be shown that

$$(52) \qquad \frac{MM_3}{I} = -\rho + \rho^2 - \rho^3 .$$

This result is consistent with the property of being non-increasing in prices in (45) since it is clear that MM_3 is positive (negative) if ρ is negative (positive). It remains to be shown what happens if all prices move in the same direction but not necessarily in the same proportion.

Finally, in the case where all prices and income change in the same proportion given by

$$(53) \qquad \tau = \frac{\Delta I}{I} = \frac{\Delta p_i}{p_i} \quad , \quad i = 1, 2, ..., n$$

it can be verified that by substitution of (53) into (48) and using the budget identities in (50),

$$(54) \qquad \frac{MM_3}{I} = 0 \ , \quad all \quad \tau \ .$$

This result implies that (48) is homogeneous of degree zero, i.e. welfare is unchanged by a proportional change in prices and income.

The properties of the money metric of being non-decreasing in income and homogeneous of degree zero in income and prices are not dependent on the order of the Taylor series approximation, at least up to a third-order as in the case of (48). However, the property of being non-increasing in prices is violated by a second-order approximation.

The first-order approximation includes only the first two terms on the right-hand side of (48). Denoting this by $\frac{MM_1}{I}$, the first-order approximation is, therefore,

$$\text{(55)} \qquad \frac{MM_1}{I} = \frac{\Delta I}{I} - \sum_i \omega_i \frac{\Delta p_i}{p_i} \ .$$

It is straightforward to show using earlier procedures that this is non-decreasing in income, non-increasing in prices and homogeneous of degree zero in income and prices.

The second-order approximation denoted by $\frac{MM_2}{I}$ consists of the first four terms of (48), i.e.,

$$\text{(56)} \qquad \frac{MM_2}{I} = \frac{\Delta I}{I} - \sum_i \omega_i \frac{\Delta p_i}{p_i} - \sum_i \omega_i \frac{I}{x_i} \frac{\partial x_i}{\partial I} \frac{\Delta p_i}{p_i} \frac{\Delta I}{I}$$
$$+ \frac{1}{2} \sum_i \sum_j \left\{ \omega_i \omega_j \frac{I}{x_j} \frac{\partial x_j}{\partial I} - \frac{p_i p_j}{I} \frac{\partial x_i}{\partial p_j} \right\} \frac{\Delta p_i}{p_i} \frac{\Delta p_j}{p_j} \ .$$

Assume now the case in (51) at the same time that income remains the same. Again using the budget identities in (50), it can be verified that

$$\text{(57)} \qquad \frac{MM_2}{I} = - \rho + \rho^2 \ .$$

This is obviously a perverse result. It is sufficient to point out that if ρ is greater than one (prices increase), that $\frac{MM_2}{I}$ is positive. This means that it is possible for welfare to improve when prices rise while income remains the same, which is nonsense.

The above result is interesting in light of the fact that, as noted by McKenzie, "Hicks chose to work with second-order Taylor series approximations to both the equivalent and compensating variations."[35] Except for the possibility of a perverse result from (57), which suffices to invalidate a second-order approximation, the

[35]See McKenzie op. cit., p. 114.

reason for its appeal appears to be that it allows for computation not only of the equivalent variation but also of the compensating variation as well. Consider again (56). By definition of the money metric, the second-order MM_2 also measures the equivalent variation from a change in prices given that income remains the same. Note, however, that if prices change and utility is held constant, i.e., $MM_2 = 0$, then $\frac{\Delta I}{I}$ can be solved from (56). But since I is initial income, ΔI can be determined, which is conceptually the compensating variation for the change in prices. This solution for the compensating variation is unique from (56) because it is linear in ΔI. However, this solution is logically inadmissible because of the perversity from (56) implied by (57). In contrast, (48) does not have the above perverse property but it does not yield a unique solution to the compensating variation for the simple reason that it is quadratic in ΔI.

The conclusion is that the money metric from a finite Taylor series approximation is not per se a welfare indicator for it depends on the order of the approximation. It was shown, for example, that a second-order approximation is not a welfare indicator since it is not non-increasing in prices. However, the third-order money metric in (48) is a welfare indicator in that it is non-decreasing in income, non-increasing in prices and homogeneous of degree zero in income and prices.

VI. A Demonstration of the Relative Accuracy of the Money Metric Over the MCS

Consider the following example that Silberberg used to demonstrate that the MCS is not uniquely measurable when the utility function is not homothetic.[36] The utility function is given by

$$U = \ln x_1 + x_2 . \tag{58}$$

[36] Silberberg op. cit., pp. 354 – 5. Note our differences in notation, which is potentially confusing. For example, he uses "log" to mean a natural logarithm as he explains in page 45. Thus, we use the usual notation "ln" in place of his "log" to mean natural logarithm. Also, we use "I" in place of his "M" to denote income. Finally, we use "MCS" in place of his "W" for the Marshallian consumer's surplus.

By maximizing (58) subject to the budget constraint

$$I = p_1 x_1 + p_2 x_2 \tag{59}$$

the resulting Marshallian demand functions are

$$x_1 = \frac{p_2}{p_1} \qquad and \tag{60}$$

$$x_2 = \left(\frac{I}{p_2} - 1\right) . \tag{61}$$

It is obvious from (60) and (61), respectively, that the income elasticity of demand for x_1 is zero and that for x_2 is positive but varies with the level of income (for I greater than p_2). Thus, the utility function is not homothetic. In this case, it will be shown that the MCS is not unique if both prices change.

Following (15), the MCS can be written generally as

$$MCS = - \int \frac{p_2}{p_1} dp_1 - \int \left(\frac{I}{p_2} - 1\right) dp_2 . \tag{62}$$

However, (62) can be specified for each possible path of integration. For this purpose, let the old (denoted by "o") and new (denoted by "n") price vectors be given, respectively, by

$$P^o = (p_1^o, p_2^o) \quad ; \quad and$$
$$P^n = (p_1^n, p_2^n) \ .$$

In the following example, it is assumed that the level of income remains the same since only the welfare effects of price changes are analyzed. That is,

$$I^o = I^n \ .$$

Since there are two prices, both of which are assumed to change, MCS will have two values for every pair of prices corresponding to the two paths of integration. These are given below. Path (1): Hold p_2 constant first, while p_1 is changing and then let p_2 change. That is, let $p_2 = p_2^o$, the old level, and let p_1 change to p_1^n, the new level. Afterwards, let p_2 change to p_2^n. Thus,

$$MCS_1 = - \int_{p_1^o}^{p_1^n} \frac{p_2^o}{p_1} dp_1 - \int_{p_2^o}^{p_2^n} \left(\frac{I^o}{p_2} - 1\right) dp_2 .$$

This line integral expression for MCS_1 yields

(63) $$MCS_1 = -p_2^o \ln\left(\frac{p_1^n}{p_1^o}\right) - I^o \ln\left(\frac{p_2^n}{p_2^o}\right) + p_2^n - p_2^o .$$

Path (2): Hold p_1 constant first, while p_2 is changing and then let p_1 change. That is, let $p_1 = p_1^o$, the old level, and let p_2 change to p_2^n, the new level. Afterwards, let p_1 change to p_1^n. In this case,

$$MCS_2 = -\int_{p_2^o}^{p_2^n} \left(\frac{I^o}{p_2} - 1\right) dp_2 - \int_{p_1^o}^{p_1^n} \frac{p_2^n}{p_1} dp_1 .$$

This line integral for MCS_2 can be evaluated as

(64) $$MCS_2 = -p_2^n \ln\left(\frac{p_1^n}{p_1^o}\right) - I^o \ln\left(\frac{p_2^n}{p_2^o}\right) + p_2^n - p_2^o .$$

Given the initial level of income, I^o, utility changes from the old level, U^o, to the new level, U^n, as P^o changes to P^n. The minimum expenditure required to attain U^n at P^o will typically differ from I^o, and this difference defines the Hicksian Equivalent Variation, HEV. In general, let us define $E(U,P)$ as the minimum expenditure required to attain the utility level U given the prices defined by the vector P. Hence,

$$E(P^o,U^o) = I^o \quad ;$$

$$E(P^o,U^n) = Min\, I(P^o,U^n) \quad ; \quad and$$

(65) $$HEV = E(P^o,U^n) - E(P^o,U^o) = E(P^o,U^n) - I^o .$$

From the utility function in (58) and demand functions in (60) and (61), the indirect utility function is

(66) $$V = \ln\left(\frac{p_2}{p_1}\right) + \frac{I}{p_2} - 1$$

and consequently

(67) $$HEV = \left[\ln\left(\frac{p_2^n}{p_1^n}\right) + \frac{I^o}{p_2^n} - \ln\left(\frac{p_2^o}{p_1^o}\right)\right] p_2^o - I^o .$$

The expression for MM_3 is taken from (48). Using the initial values of [37]

$$(68) \qquad P^o = (p_1^o, p_2^o) = (2,2) \quad ; \quad I^o = 4,$$

it can be shown that MM_3 in (48) yields

$$(69) \qquad MM_3 = (\Delta p_1 + \Delta p_2)\left[\frac{1}{4}\Delta p_2 - \frac{1}{12}(\Delta p_2)^2 - 1\right] + \frac{1}{4}(\Delta p_1)^2$$

$$- \frac{1}{4}\Delta p_1\,\Delta p_2 + \frac{1}{2}(\Delta p_2)^2 + \frac{1}{12}\Delta p_1(\Delta p_2)^2 - \frac{1}{3}(\Delta p_2)^3 - \frac{1}{12}(\Delta p_1)^3$$

where $\Delta p_1 = p_1^n - p_1^o$ and $\Delta p_2 = p_2^n - p_2^o$.

The values of HEV in (67) are summarized in Table 1 for a range of alternative prices. These values range widely from an increase of over 50 % in real income to a decrease of over 25 %, given the initial prices and income in (68).

Table 1. Values of HEV at Alternative Prices

$p_1^n \Downarrow p_2^n \Rightarrow$	1.25	1.50	1.75	2.00	2.25	2.50	2.75
1.25	2.400	1.698	1.244	0.940	0.731	0.586	0.486
1.50	2.035	1.333	0.880	0.575	0.366	0.222	0.121
1.75	1.727	1.025	0.571	0.267	0.058	−0.087	−0.187
2.00	1.460	0.758	0.304	0.000	−0.209	−0.354	−0.454
2.25	1.224	0.522	0.069	−0.236	−0.444	−0.589	−0.690
2.50	1.014	0.312	−0.142	−0.446	−0.655	−0.800	−0.900
2.75	0.823	0.121	−0.333	−0.637	−0.846	−0.991	−1.091

[37] These are the same as those in Silberberg's example.

Table 2. Percent Ratios of MCS_1, MCS_2 and MM_3 to HEV

$p_1^n \Downarrow$	$p_2^n \Rightarrow$	1.25	1.50	1.75	2.00	2.25	2.50	2.75
1.25	$\frac{MCS_1}{HEV}$	86.25	93.68	98.37	100.00	98.32	93.37	85.64
	$\frac{MCS_2}{HEV}$	71.56	79.84	88.93	100.00	114.40	133.45	158.17
	$\frac{MM_3}{HEV}$	94.73	98.08	98.78	98.49	97.95	95.72	86.80
1.50	$\frac{MCS_1}{HEV}$	83.79	91.96	97.70	100.00	96.65	82.47	42.48
	$\frac{MCS_2}{HEV}$	73.19	81.17	89.52	100.00	116.28	147.36	220.26
	$\frac{MM_3}{HEV}$	94.36	98.44	99.61	99.57	99.13	94.00	56.86
1.75	$\frac{MCS_1}{HEV}$	80.89	89.54	96.46	100.00	78.94	144.85	137.35
	$\frac{MCS_2}{HEV}$	75.09	83.02	90.62	100.00	136.31	67.80	83.77
	$\frac{MM_3}{HEV}$	93.49	98.19	99.80	99.95	98.47	112.70	126.77
2.00	$\frac{MCS_1}{HEV}$	77.40	85.85	93.35	—	105.87	110.99	115.38
	$\frac{MCS_2}{HEV}$	77.40	85.85	93.35	—	105.87	110.99	115.38
	$\frac{MM_3}{HEV}$	92.31	97.57	99.68	—	100.36	103.07	111.00
2.25	$\frac{MCS_1}{HEV}$	73.05	79.47	70.58	100.00	102.76	106.59	110.12
	$\frac{MCS_2}{HEV}$	80.26	90.74	113.38	100.00	109.38	116.59	122.93
	$\frac{MM_3}{HEV}$	90.82	96.46	98.41	100.05	100.20	101.86	107.25
2.50	$\frac{MCS_1}{HEV}$	67.45	65.59	114.26	100.0	101.87	104.86	107.75
	$\frac{MCS_2}{HEV}$	83.96	101.39	74.95	100.00	110.39	118.80	126.34
	$\frac{MM_3}{HEV}$	88.76	93.58	101.84	100.37	100.36	101.56	105.72
2.75	$\frac{MCS_1}{HEV}$	59.91	11.42	106.09	100.00	101.45	103.92	106.40
	$\frac{MCS_2}{HEV}$	88.93	142.94	82.15	100.00	110.86	120.00	128.29
	$\frac{MM_3}{HEV}$	85.43	78.52	102.59	101.20	101.0	101.87	105.27

In Table 2, the two MCS measures and MM_3 are compared with the HEV values in Table 1. In all cases, the direction of change is correct for all three measures. However, the errors are generally much larger for the MCS measures than for MM_3. The errors range from -89 % to $+45$ %, -25 % to $+120$ % for MCS_1 and MCS_2, respectively, compared to -45 % to $+27$ % for

MM_3. Both of the extreme values for MM_3 correspond to the largest level of p_2 where the values of HEV are relatively small. In fact, 82 % of the values of MM_3 are within 10 % of the true value compared to only 51 % and 21 % for MCS_1 and MCS_2, respectively. It should be noted that the use of the averages of the two different MCS measures, given the same combination of prices, is not a practical way to improve accuracy. This is because, in more general situations with many price changes and unequal income elasticities, the number of possible MCS values (for the same combination of prices) is very large corresponding to a large number of possible paths of integration. This latter possibility underscores the futility of the approximations suggested by Willig. [38] Averaging will simply mask the fact that, in the above situations, the MCS will yield dramatically different measures of welfare change for the same change in prices.

VII. Qualifications and Conclusions

The framework of this paper is the measurement of the effects on welfare of a simultaneous change in prices and income in a demand system with unequal income elasticities. The findings of this paper, as well as those in the literature, leads to the conclusion that the conditions for the integrability or unique measurability of MCS are theoretically so restrictive as to render it unworkable in practice. In particular, the necessary and sufficient condition for its integrability that the income elasticities of the goods with changing prices are uniformly equal (not necessarily to one or zero) faces overwhelming evidence to the contrary.[39] This empirical evidence implies that the MCS is most likely to be very inaccurate in practice.

[38] Willig,op. cit..

[39] See, for example, Deaton, A. and Muellbauer, J., Economics and Consumer Behavior (Cambridge: Cambridge University Press, 1984); Deaton, A. and Muellbauer, J., "An Almost Ideal Demand System," American Economic Review (vol. 70., n0. 3, June 1980), pp. 312 – 26.; Johnson, S. R., Hassan, Z. A. and Green, R. D., Demand Systems Estimation, Methods and Applications (Ames: Iowa State University Press, 1984); and Phlips, L., Applied Consumption Analysis (Amsterdam: North-Holland Publishing Co., 1974) for a compendium of references.

In the more general case of unequal income elasticities, the only theoretically meaningful measure of welfare change is the HEV. It has been established in theory that an income compensation function equivalent to the HEV exists and what is lacking is a practical algorithm to compute it from observable Marshallian demand functions. For this purpose, McKenzie has proposed a money metric to measure the HEV from these demand functions. This paper shows that a third-order money metric satisfies the integrability conditions and is applicable to the more general case of unequal income elasticities.

This paper has reexamined the money metric as a finite order Taylor series approximation to an unknown indirect utility function. The results show that the money metric is not a welfare indicator per se because it does not necessarily possess all the essential economic properties of an indirect utility function, namely, non-decreasing in income, non-increasing in prices and homogeneous of degree zero in income and prices. In particular, a second-order approximation is invalid since it is not non-increasing in prices. However, the money metric from a third-order approximation possesses all the above three properties and is, therefore, a valid measure of welfare change.

This paper tested the reformulation of the third-order money metric against the MCS as approximations to the true HEV. All measures determined the direction of change correctly in all cases. By construction of the example, the MCS was not unique and the two possible MCS measures were often dramatically different from each other. The results show that the money metric is a far more accurate measure of the HEV than the MCS, especially in cases where the price changes are less than 50%.

The money metric does have the drawback of being inexact because it is a finite order Taylor series approximation. Hence, it is subject to the error of ignoring the remainder term. In this respect, McKenzie observed that: "Unless we know exactly the function that we are attempting to approximate, it is impossible to establish convincing error bounds or regions of ignorance."[40]

Unfortunately, the approximation error above cannot be solved, in general,

[40]McKenzie, op. cit., p. 116.

when measuring the welfare change from an unknown indirect utility function. Given the Marshallian demand functions, this change can be precisely measured by the MCS if and only if the income elasticities are uniformly equal for the goods with changing prices. Without this condition, the MCS simply does not work because it is not uniquely measurable. As demonstrated in the example, any one of the MCS measures is only an approximation subject to errors that were shown to be very large indeed. In the general case of unequal income elasticities, where HEV is the theoretically meaningful money equivalent of a change in welfare, any alternative measure could only be an approximation because of our ignorance of the exact form of the underlying utility function. Thus, with respect to the money metric, the issue is not that it is an approximation, but rather whether or not there is a better measure of HEV.

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